CK-12

flexbook
next generation textbooks

CK-12 FlexBook

MCPS C2.0 Geometry Unit 2 Topic 2 Flexbook

MCPS Math
MCPS

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

AUTHORS
MCPS Math
MCPS

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-content, web-based collaborative model termed the **FlexBook®**, CK-12 intends to pioneer the generation and distribution of high-quality educational content that will serve both as core text as well as provide an adaptive environment for learning, powered through the **FlexBook Platform®**.

Printed: November 6, 2014

Contents

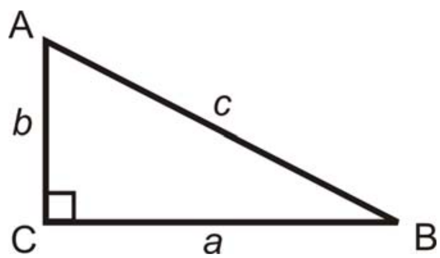
CONCEPT 1 SLT 16 Explore the relationships between sides and angles in a right triangle.

What if you were given a right triangle? What are the relationships between the sides and angles?

Guidance

The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define three trigonometric (or trig) functions.

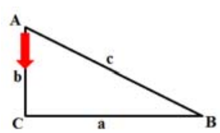
Trigonometry: The study of the relationships between the sides and angles of right triangles.



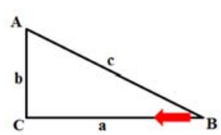
In trigonometry, sides are named in reference to a particular angle. The hypotenuse, c, of a triangle is always the same, but the terms **adjacent** and **opposite** depend on which angle you are referencing.

A side **adjacent** to an angle is the leg of the triangle that helps form the angle.

In the right triangle below, side b is adjacent to angle A.

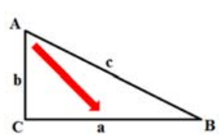


In the right triangle below, side a is adjacent to angle B.

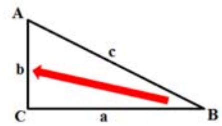


A side **opposite** to an angle is the leg of the triangle that does not help form the angle.

In the right triangle below, side a is opposite angle A.

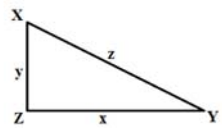


In the right triangle below, side b is opposite angle B.



Example A

Given triangle XYZ,



1. What side is opposite angle X?
2. What side is adjacent angle Y?

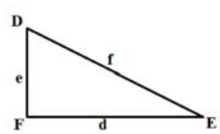
Answers:

1. x
2. x

Example B

Draw right triangle DEF, such that angle E is adjacent to side d and opposite side e.

Answer:

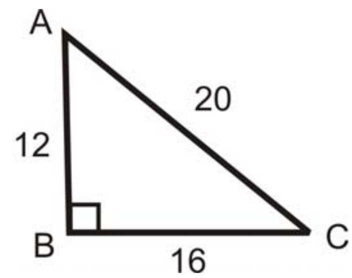


Vocabulary

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being referenced.

Guided Practice

Answer the questions about the following image. Reduce all fractions.



1. What is the measure of the side that is opposite angle C?
2. What is the measure of the side that is adjacent angle C?
3. What side measures 20 units?

Answers:

1. 12
2. 16
3. the hypotenuse

Explore More

Use the diagram to fill in the blanks below.

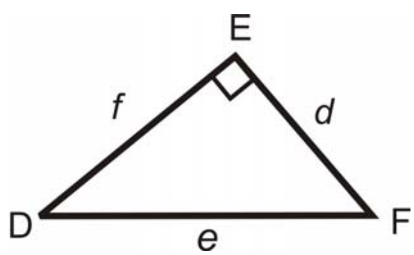


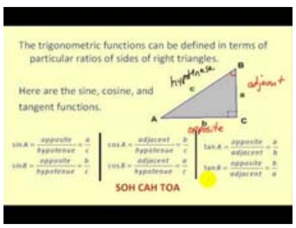
TABLE 1.1:

Angle	Opposite side	Adjacent side
1. D	2.	3.
4.	5.	6. d

CONCEPT 2 SLTs 17 & 18 Find triangle side ratios for sine, cosine, tangent, cosecant, secant, and cotangent.

Trigonometry is the study of triangles. There are six trigonometric ratios which relate the angles and sides of right triangles.

Watch This



MEDIA

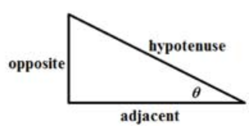
Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58116

http://www.youtube.com/watch?v=Ujyl_zQw2zE James Sousa: Introduction to Trigonometric Functions Using Triangles

Guidance

The six trigonometric ratios of the angle θ are sine, cosine, tangent, cotangent, secant and cosecant.



$\sin\theta = \frac{opposite}{hypotenuse}$

$\cos\theta = \frac{adjacent}{hypotenuse}$

$\tan\theta = \frac{opposite}{adjacent}$

$\cot\theta = \frac{adjacent}{opposite}$

$\sec\theta = \frac{hypotenuse}{adjacent}$

$\csc\theta = \frac{hypotenuse}{opposite}$

The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. There is a reciprocal relationship between them.

$\csc\theta = \frac{hypotenuse}{opposite} = \frac{1}{\sin\theta}$

$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{1}{\cos\theta}$

$\cot\theta = \frac{adjacent}{opposite} = \frac{1}{\tan\theta}$

Note: The images throughout this concept are not drawn to scale.

Example A

Find the sine, cosine and tangent ratios of $\angle A$.

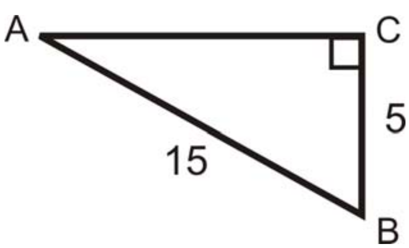
First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

$$5^2 + 12^2 = h^2$$
$$13 = h$$

So, $\sin A = \frac{12}{13}, \cos A = \frac{5}{13}$, and $\tan A = \frac{12}{5}$.

Example B

Find the sine, cosine, and tangent of $\angle B$.



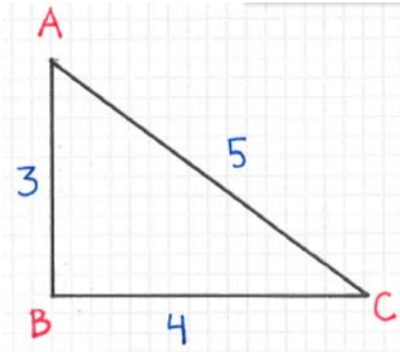
Find the length of the missing side.

$$AC^2 + 5^2 = 15^2$$
$$AC^2 = 200$$
$$AC = 10\sqrt{2}$$

Therefore, $\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}, \cos B = \frac{5}{15} = \frac{1}{3}$, and $\tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$.

Example C

List the six trigonometric ratios for the non-right angles in the triangle.



$\sin A = \frac{4}{5}, \cos A = \frac{3}{5}, \tan A = \frac{4}{3}, \csc A = \frac{5}{4}, \sec A = \frac{5}{3}, and \cot A = \frac{3}{4}$

$\sin B = \frac{3}{5}, \cos B = \frac{4}{5}, \tan B = \frac{3}{4}, \csc B = \frac{5}{3}, \sec B = \frac{5}{4}, and \cot B = \frac{4}{3}$

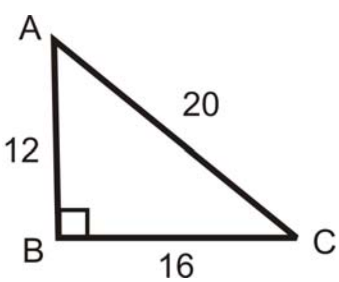
Vocabulary

The ***six trigonometric ratios*** are universal proportions that are always true of similar triangles (triangles with congruent corresponding angles).

θ ***(theta)*** is a Greek letter and is just a letter used in math to stand for an unknown angle.

Guided Practice

Answer the questions about the following image.



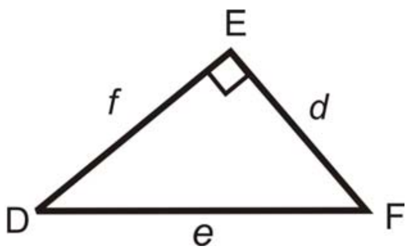
1. What is $\sin A$?
2. What is $\cos A$?
3. What is $\tan A$?

Answers:

1. $\sin A = \frac{16}{20} = \frac{4}{5}$
2. $\cos A = \frac{12}{20} = \frac{3}{5}$
3. $\tan A = \frac{16}{12} = \frac{4}{3}$

Explore more

Use the diagram to fill in the blanks below.

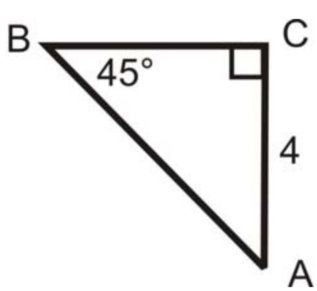


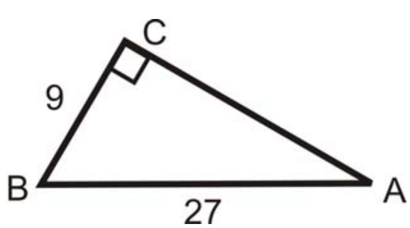
1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\sec F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\csc D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

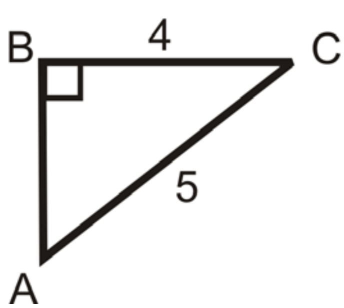
From questions 1-6, we can conclude the following. Fill in the blanks.

7. $\cos$___ $= \sin F$ and $\sin$___ $= \cos F$.
8. $\tan D$ and $\tan F$ are __________ of each other.

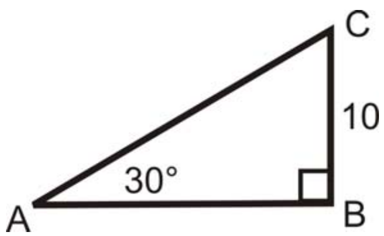
Find the six trigonometric ratios of $\angle A$.

9.


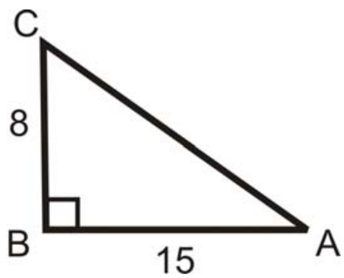
10.


11.


12.



13.



14. Explain why the sine of an angle will never be greater than 1.
15. Explain why the tangent of a 45° angle will always be 1.
16. As the degree of an angle increases, will the tangent of the angle increase or decrease? Explain.

CONCEPT 3 SLT 19 Explore the connection between trigonometric ratios and their associated angle.

What if you wanted to find the missing sides of a right triangle with angles of 20° and 70° and a hypotenuse length of 10 inches? How could you use trigonometry to help you?

Watch This

Multimedia

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52511

CK-12 Foundation: Chapter8TrigonometricRatioswithaCalculatorA

Guidance

The trigonometric ratios are not dependent on the exact side lengths, but the angles. There is one fixed value for every angle, from 0° to 90°. Your scientific (or graphing) calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS], and [TAN] buttons. Use these to find the sine, cosine, and tangent of any acute angle. One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side.

Example A

Find the trigonometric value, using your calculator. Round to 4 decimal places.

a) $\sin 78^\circ$

b) $\cos 60^\circ$

c) $\tan 15^\circ$

Depending on your calculator, you enter the degree and then press the trig button or the other way around. Also, make sure the mode of your calculator is in *DEGREES*.

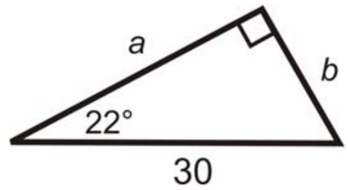
a) $\sin 78^\circ = 0.97815$

b) $\cos 60^\circ = 0.5$

c) $\tan 15^\circ = 0.26795$

Example B

Find the value of each variable. Round your answer to the nearest tenth.

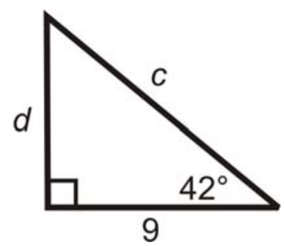


We are given the hypotenuse. Use *sine* to find b, and *cosine* to find a. Use your calculator to evaluate the sine and cosine of the angles.

$$\begin{aligned} \sin 22^\circ &= \frac{b}{30} \\ 30 \cdot \sin 22^\circ &= b \\ b &\approx 11.2 \end{aligned} \qquad \begin{aligned} \cos 22^\circ &= \frac{a}{30} \\ 30 \cdot \cos 22^\circ &= a \\ a &\approx 27.8 \end{aligned}$$

Example C

Find the value of each variable. Round your answer to the nearest tenth.



We are given the adjacent leg to 42°. To find c, use *cosine* and use *tangent* to find d.

$$\begin{aligned} \cos 42^\circ &= \frac{adjacent}{hypotenuse} = \frac{9}{c} \\ c \cdot \cos 42^\circ &= 9 \\ c &= \frac{9}{\cos 42^\circ} \approx 12.1 \end{aligned} \qquad \begin{aligned} \tan 42^\circ &= \frac{opposite}{adjacent} = \frac{d}{9} \\ 9 \cdot \tan 42^\circ &= d \\ d &\approx 27.0 \end{aligned}$$

Any time you use trigonometric ratios, use only the information that you are given. This will result in the most accurate answers.

Watch this video for help with the Examples above.

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/52512

CK-12 Foundation: Chapter8TrigonometricRatioswithaCalculatorB

Concept Problem Revisited

Use trigonometric ratios to find the missing sides. Round to the nearest tenth.

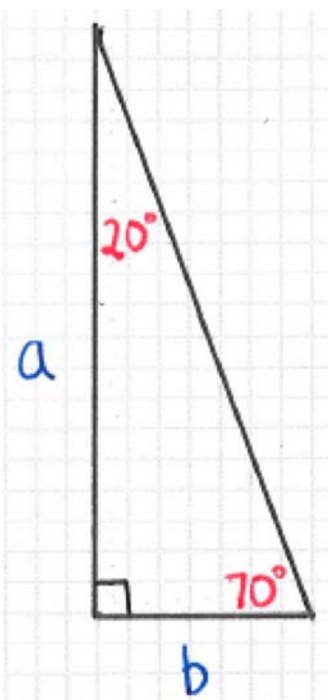


FIGURE 3.1

Find the length of a and b using sine or cosine ratios:

$$\begin{aligned} \cos 20^\circ &= \frac{a}{10} \\ 10 \cdot \cos 20^\circ &= a \\ a &\approx 9.4 \end{aligned} \qquad \begin{aligned} \sin 70^\circ &= \frac{a}{10} \\ 10 \cdot \sin 70^\circ &= a \\ a &\approx 9.4 \end{aligned}$$

$$\begin{aligned} \sin 20^\circ &= \frac{b}{10} \\ 10 \cdot \sin 20^\circ &= b \\ b &\approx 3.4 \end{aligned} \qquad \begin{aligned} \cos 70^\circ &= \frac{b}{10} \\ 10 \cdot \cos 70^\circ &= b \\ b &\approx 3.4 \end{aligned}$$

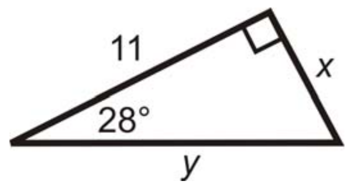
Vocabulary

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used. The three trigonometric (or trig) ratios are ***sine***, ***cosine***, and ***tangent***.

Guided Practice

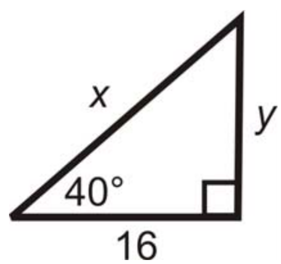
1. What is $\tan 45^\circ$?

2. Find the length of the missing sides and round your answers to the nearest tenth:



.

3. Find the length of the missing sides and round your answers to the nearest tenth:



.

Answers:

1. Using your calculator, you should find that $\tan 45^\circ = 1$?

2. Use tangent for x and cosine for y.

$$\tan 28^\circ = \frac{x}{11} \qquad \cos 28^\circ = \frac{11}{y}$$
$$11 \cdot \tan 28^\circ = x \qquad \frac{11}{\cos 28^\circ} = y$$
$$x \approx 5.8 \qquad y \approx 12.5$$

3. Use tangent for y and cosine for x.

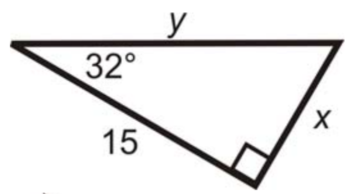
$$\tan 40^\circ = \frac{y}{16} \qquad \cos 40^\circ = \frac{16}{x}$$
$$16 \cdot \tan 40^\circ = y \qquad \frac{16}{\cos 40^\circ} = x$$
$$y \approx 13.4 \qquad x \approx 20.9$$

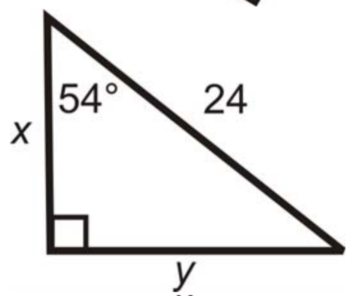
Explore More

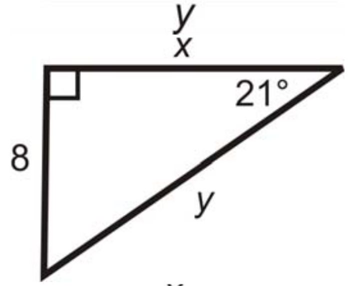
Use your calculator to find the value of each trig function below. Round to four decimal places.

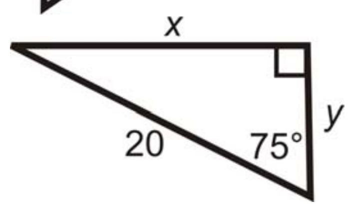
1. $\sin 24^\circ$
2. $\cos 45^\circ$
3. $\tan 88^\circ$
4. $\sin 43^\circ$
5. $\tan 12^\circ$
6. $\cos 79^\circ$
7. $\sin 82^\circ$

Find the length of the missing sides. Round your answers to the nearest tenth.

8. 

9. 

10. 

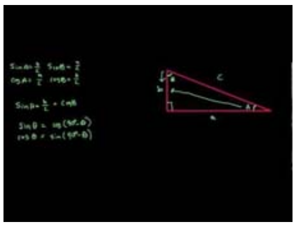
11. 

12. Find $\sin 80^\circ$ and $\cos 10^\circ$.
13. Use your knowledge of where the trigonometric ratios come from to explain your result to the previous question.
14. Generalize your result to the previous two questions. If $\sin \theta = x$, then $\cos ? = x$.
15. How are $\tan \theta$ and $\tan(90 - \theta)$ related? Explain.

CONCEPT 4 SLT 20 Determine the relationship between sine and cosine.

ΔABC is a right triangle with $m\angle C = 90^\circ$ and $\sin A = k$. What is $\cos B$?

Watch This



MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/75771

http://www.youtube.com/watch?v=gEPMmcDJSB8

Guidance

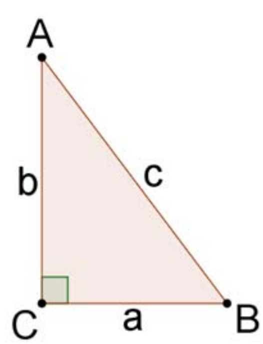
Recall that the **sine** and **cosine** of angles are ratios of pairs of sides in right triangles.

- The **sine** of an angle in a right triangle is the ratio of the side *opposite* the angle to the *hypotenuse.*
- The **cosine** of an angle in a right triangle is the ratio of the side *adjacent* to the angle to the *hypotenuse*.

In the examples, you will explore how the sine and cosine of the angles in a right triangle are related.

Example A

Consider the right triangle below. Find the sine and cosine of angles A and B in terms of a, b, and c. What do you notice?



Solution: $\sin A = \frac{a}{c}, \sin B = \frac{b}{c}, \cos A = \frac{b}{c}, \cos B = \frac{a}{c}$. Note that $\sin A = \cos B$ and $\sin B = \cos A$.

Example B

Consider the triangle from Example A. How is $\angle A$ related to $\angle B$?

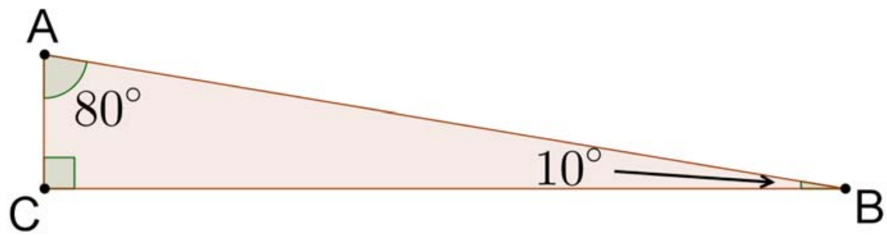
Solution: The sum of the measures of the three angles in a triangle is 180°. This means that $m\angle A + m\angle B + m\angle C = 180^\circ$. $\angle C$ is a right angle so $m\angle C = 90^\circ$. Therefore, $m\angle A + m\angle B = 90^\circ$. Angles A and B are complementary angles because their sum is 90°.

In Example A you saw that $\sin A = \cos B$ and $\sin B = \cos A$. This means that the sine and cosine of *complementary angles* are equal.

Example C

Find 80° and $\cos 10^\circ$. Explain the result.

Solution: $\sin 80^\circ \approx 0.985$ and $\cos 10^\circ \approx 0.985$. $\sin 80^\circ = \cos 10^\circ$ because 80° and 10° are complementary angle measures. $\sin 80^\circ$ and $\cos 10^\circ$ are the ratios of the same sides of a right triangle, as shown below.



Concept Problem Revisited

ΔABC is a right triangle with $m\angle C = 90^\circ$ and $\sin A = k$. What is $\cos B$?

$\angle A$ and $\angle B$ are complementary because they are the two non-right angles of a right triangle. This means that $\sin A = \cos B$ and $\sin B = \cos A$. If $\sin A = k$, then $\cos B = k$ as well.

Vocabulary

The ***tangent (tan)*** of an angle within a right triangle is the ratio of the length of the side opposite the angle to the length of the side adjacent to the angle.

The ***sine (sin)*** of an angle within a right triangle is the ratio of the length of the side opposite the angle to the length of the hypotenuse.

The ***cosine (cos)*** of an angle within a right triangle is the ratio of the length of the side adjacent to the angle to the length of the hypotenuse.

The ***trigonometric ratios*** are sine, cosine, and tangent.

Trigonometry is the study of triangles.

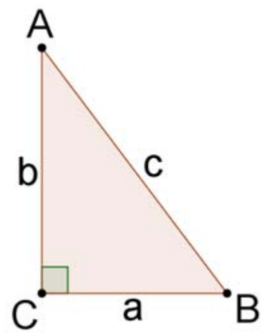
θ, or ***"theta",*** is a Greek letter. In geometry, it is often used as a variable to represent an unknown angle measure.

Two angles are ***complementary*** if the sum of their measures is 90°.

Guided Practice

1. If $\sin 30^\circ = \frac{1}{2}, \cos ? = \frac{1}{2}$.

2. Consider the right triangle below. Find $\tan A$ and $\tan B$.



3. In general, what is the relationship between the tangents of complementary angles?

Answers:

1. The sine and cosine of **complementary** angles are equal. $90^\circ - 30^\circ = 60^\circ$ is complementary to 30°. Therefore, $\cos 60^\circ = \frac{1}{2}$.

2. $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

3. In general, the tangents of complementary angles are reciprocals.

Practice

1. How are the two non-right angles in a right triangle related? Explain.
2. How are the sine and cosine of complementary angles related? Explain.
3. How are the tangents of complementary angles related? Explain.

Let A and B be the two non-right angles in a right triangle.

4. If $\tan A = \frac{1}{2}$, what is $\tan B$?
5. If $\sin A = \frac{7}{10}$, what is $\cos B$?
6. If $\cos A = \frac{1}{4}$ what is $\sin B$?
7. If $\sin A = \frac{3}{5}, \cos ? = \frac{3}{5}$?
8. Simplify $\frac{\sin A + \cos B}{2}$.
9. If $\tan A = \frac{2}{3}$ what is $\tan B$?
10. If $\tan B = \frac{1}{5}$, what is $\tan A$? Which angle is bigger, $\angle A$ or $\angle B$?

Solve for θ.

11. $\cos 30^\circ = \sin \theta$
12. $\sin 75^\circ = \cos \theta$
13. $\cos 52^\circ = \sin \theta$
14. $\sin 18^\circ = \cos \theta$
15. $\cos 49^\circ = \sin \theta$

References

1. . . CC BY-NC-SA
2. . . CC BY-NC-SA
3. . . CC BY-NC-SA

CONCEPT 5 SLT 21 Apply trigonometric ratios to solve for missing angles and sides of right triangles.

What if you were told that the longest escalator in North America is at the Wheaton Metro Station in Maryland and is 230 feet long and is 115 ft high? What is the angle of elevation, x°, of this escalator?

Watch This

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52521

CK-12 Foundation: Chapter8InverseTrigonometricRatiosA

Guidance

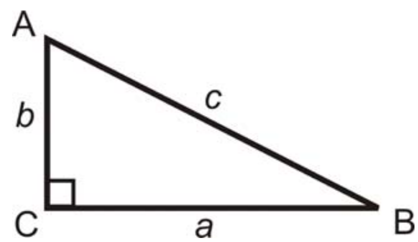
The word ***inverse*** is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to "undo" it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities. When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. With the inverse trig ratios, you can find the angle measure, given two sides.

Inverse Tangent: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle. Inverse tangent is also called arctangent and is labeled $\tan^{-1}$ or *arctan*. The "-1" indicates inverse.

Inverse Sine: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle. Inverse sine is also called arcsine and is labeled $\sin^{-1}$ or *arcsin*.

Inverse Cosine: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle. Inverse cosine is also called arccosine and is labeled $\cos^{-1}$ or *arccos*.

Using the triangle below, the inverse trigonometric ratios look like this:



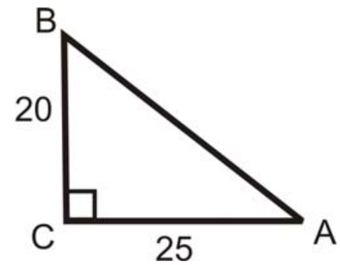
$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B \qquad \tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$
$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B \qquad \sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$
$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B \qquad \cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to actually find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like $[\text{SIN}^{-1}], [\text{COS}^{-1}]$, and $[\text{TAN}^{-1}]$. Typically, you might have to hit a shift or 2^{nd} button to access these functions. For example, on the TI-83 and 84, $[2^{nd}][\text{SIN}]$ is $[\text{SIN}^{-1}]$. Again, make sure the mode is in degrees.

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles to only use the values that you are given.

Example A

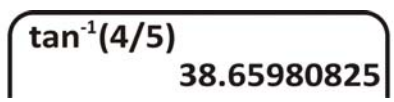
Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$, therefore $\tan^{-1}\left(\frac{4}{5}\right) = m\angle A$. Use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}][\text{TAN}]\left(\frac{4}{5}\right)$ **[ENTER]** and the screen looks like:



So, $m\angle A = 38.7^\circ$

Example B

$\angle A$ is an acute angle in a right triangle. Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solutions:

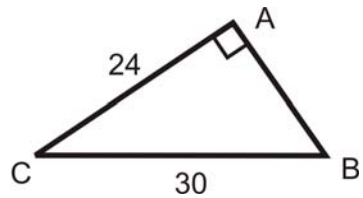
a) $m\angle A = \sin^{-1} 0.68 = 42.8^\circ$

b) $m\angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m\angle A = \tan^{-1} 0.34 = 18.8^\circ$

Example C

Solve the right triangle.



To solve this right triangle, we need to find $AB, m\angle C$ and $m\angle B$. Use AC and CB to give the most accurate answers.

$\underline{AB}$: Use the Pythagorean Theorem.

$$\begin{aligned}24^2 + AB^2 &= 30^2\\ 576 + AB^2 &= 900\\ AB^2 &= 324\\ AB &= \sqrt{324} = 18\end{aligned}$$

$\underline{m\angle B}$: Use the inverse sine ratio.

$$\begin{aligned}\sin B &= \frac{24}{30} = \frac{4}{5}\\ \sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B\end{aligned}$$

$\underline{m\angle C}$: Use the inverse cosine ratio.

$$\begin{aligned}\cos C &= \frac{24}{30} = \frac{4}{5}\\ \cos^{-1}\left(\frac{4}{5}\right) &= 36.9^\circ = m\angle C\end{aligned}$$

Watch this video for help with the Examples above.

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52522

CK-12 Foundation: Chapter8InverseTrigonometricRatiosB

Concept Problem Revisited

To find the escalator's angle of elevation, we need to use the inverse sine ratio.

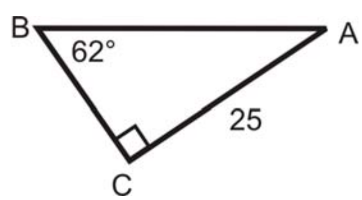
$$\sin^{-1}\left(\frac{115}{230}\right) = 30^\circ \qquad \text{The angle of elevation is } 30^\circ.$$

Vocabulary

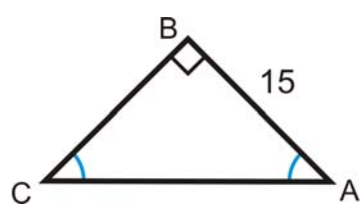
Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used. The three trigonometric (or trig) ratios are ***sine***, ***cosine***, and ***tangent***. The ***inverse*** trig ratios, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, allow us to find missing angles when we are given sides.

Guided Practice

1. Solve the right triangle.



2. Solve the right triangle.



Answers:

1. To solve this right triangle, we need to find AB, BC and $m\angle A$.

$\underline{AB}$: Use sine ratio.

$$\begin{aligned}\sin 62^\circ &= \frac{25}{AB}\\ AB &= \frac{25}{\sin 62^\circ}\\ AB &\approx 28.31\end{aligned}$$

$\underline{BC}$: Use tangent ratio.

$$\begin{aligned}\tan 62^\circ &= \frac{25}{BC}\\ BC &= \frac{25}{\tan 62^\circ}\\ BC &\approx 13.30\end{aligned}$$

$\underline{m\angle A}$: Use Triangle Sum Theorem

$$\begin{aligned}62^\circ + 90^\circ + m\angle A &= 180^\circ\\ m\angle A &= 28^\circ\end{aligned}$$

2. Even though, there are no angle measures given, we know that the two acute angles are congruent, making them both 45°. Therefore, this is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\begin{aligned}\tan 45^\circ &= \frac{15}{BC}\\ BC &= \frac{15}{\tan 45^\circ} = 15\end{aligned}$$

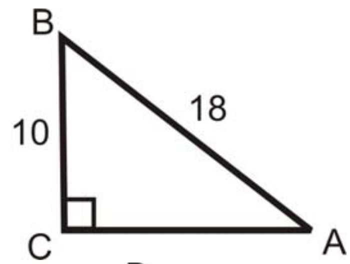
$$\begin{aligned}\sin 45^\circ &= \frac{15}{AC}\\ AC &= \frac{15}{\sin 45^\circ} \approx 21.21\end{aligned}$$

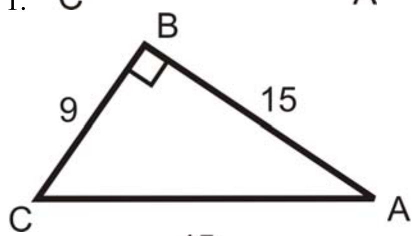
45-45-90 Triangle Ratios

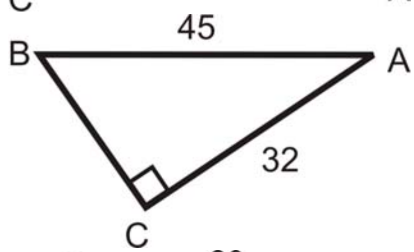
$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

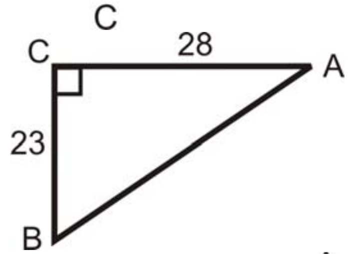
Explore More

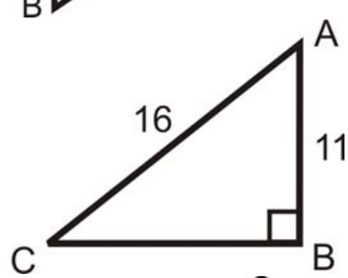
Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

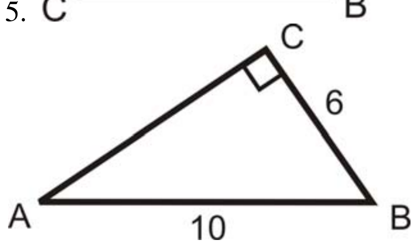
1. 

2. 

3. 

4. 

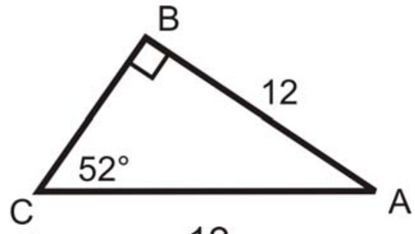
5. 

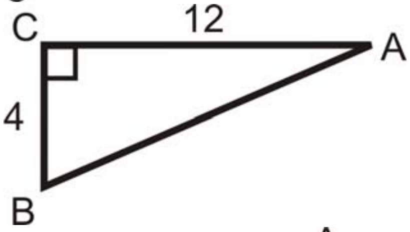
6. 

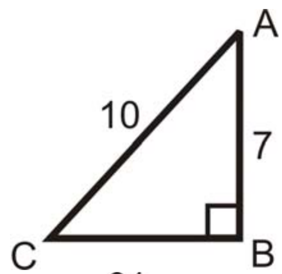
Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

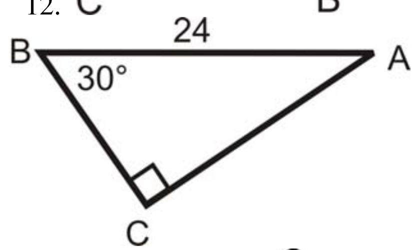
7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$

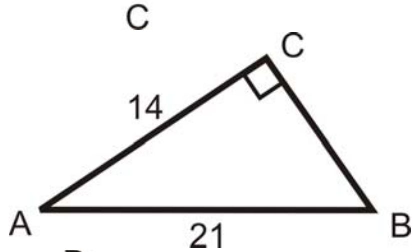
Solve the following right triangles. Find all missing sides and angles.

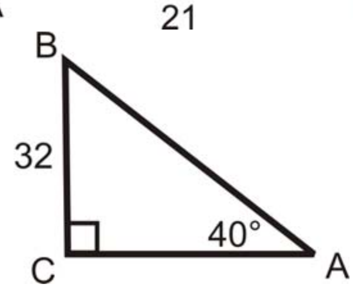
10. 

11. 

12. 

13. 

14. 

15. 

16. ***Writing*** Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

CONCEPT 6 SLT 22 Apply trigonometric ratios and the Pythagorean Theorem to solve for missing angles and sides.

Trigonometry is the study of triangles. If you know the measure of one angle and one side of a right triangle or the measure of two sides, you can use trigonometric ratios to completely solve for the measures of the missing sides and/or measures of missing angles.

Watch This

https://www.khanacademy.org/math/trigonometry/basic-trigonometry/basic_trig_ratios/v/example-trig-to-solve-the-sides-and-angles-of-a-right-triangle

Guidance

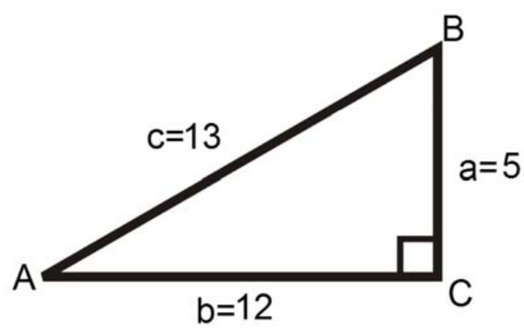
When you solve a triangle, you are finding the measures of all missing sides and all missing angles.

Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

Note: The images throughout this concept are not drawn to scale.

Example A

Given the measures of three sides of $\triangle ABC$, solve the triangle.



Solution: This problem can be solved using sin, cos or tan because the opposite, adjacent and hypotenuse lengths are all given.

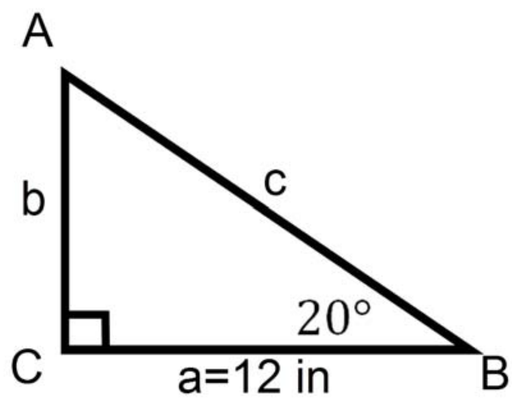
The argument of a sin function is always an angle. The arcsin or $\sin^{-1}\theta$ function on the calculator on the other hand has an argument that is a side ratio. It is useful for finding angles that have that side ratio.

$$\sin A = \frac{5}{13}$$
$$A = \sin^{-1}\left(\frac{5}{13}\right) \approx 22.6^\circ$$

We know the measure of $A = 22.6^\circ$ and the measure of $C = 90^\circ$, so the measure of $B = 180^\circ - 22.6^\circ - 90^\circ = 67.4^\circ$.

Example B

Given a right triangle with $a = 12\ in$, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, solve the triangle.



Solution:

$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77\ in.$$

We know the measure of $B = 20^\circ$ and the measure of $C = 90^\circ$, so the measure of $A = 180^\circ - 20^\circ - 90^\circ = 70^\circ$.

To find the measure of side b, we can use the Pythagorean Theorem.

$a^2 + b^2 = c^2$

$12^2 + b^2 = 12.77^2$

$144 + b^2 = 163.0729$

$b^2 = 19.0729$

$b = 4.37in.$

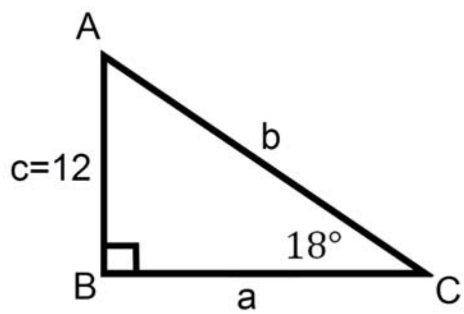
Guided Practice

1. Given $\triangle ABC$ where B is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is a?

2. Given $\triangle MNO$ where O is a right angle, $m = 12$, and $n = 14$. What is the measure of angle M?

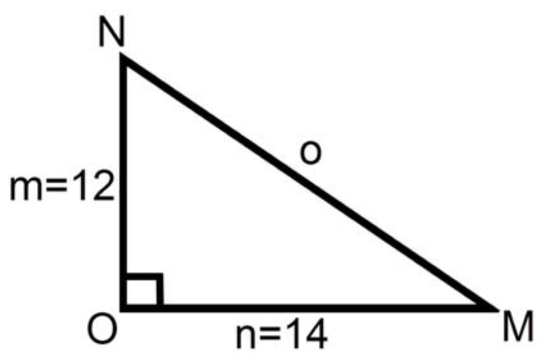
Answers:

1. Drawing out this triangle, it looks like:



$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

2. Drawing out the triangle, it looks like:



$$\tan M = \frac{12}{14}$$
$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 40.6^\circ$$

Practice

For #1-8, information about the measures of sides and/or measures of angles of right triangle *ABC* is given. Completely solve the triangle (find all missing measures of sides and angles) to 1 decimal place.

TABLE 6.1:

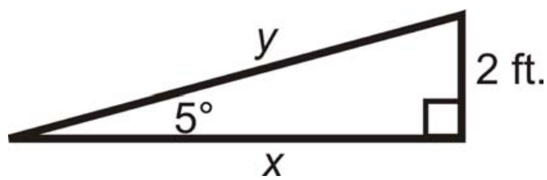
Problem Number	*A*	*B*	*C*	*a*	*b*	*c*
1.	90°				4	7
2.	90°		37°	18		
3.		90°	15°		32	
4.			90°	6		11
5.	90°	12°		19		
6.		90°			17	10
7.	90°	10°			2	
8.	4°	90°		0.3		

References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA

CONCEPT 7 SLTs 23 & 24 Model and solve application problems involving right triangles.

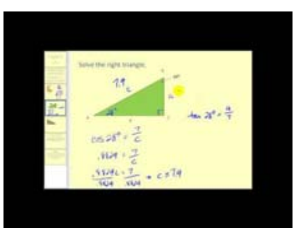
What if a restaurant needed to build a wheelchair ramp for its customers? The angle of elevation for a ramp is recommended to be 5°. If the vertical distance from the sidewalk to the front door is two feet, what is the horizontal distance that the ramp will take up (x)? How long will the ramp be (y)? Round your answers to the nearest hundredth. After completing this Concept, you'll be able to use trigonometry to solve this problem.



Watch This

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/52513

CK-12 Foundation: Chapter8TrigonometryWordProblemsA



MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1369

James Sousa: Solving Right Triangles - The Basics

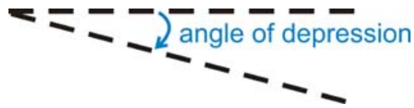
MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1370

James Sousa: Solving Right Triangles - Applications

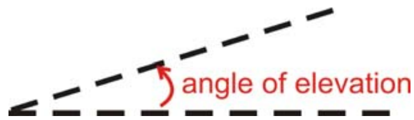
Guidance

A practical application of the trigonometric functions is to find the measure of lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.



Angle of Elevation: The angle measure from the horizon or horizontal line, up.



Example A

An inquisitive math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4°. If her "eye height" is 5ft, how tall is the monument?

Solution: We can find the height of the monument by using the tangent ratio and then adding the eye height of the student.

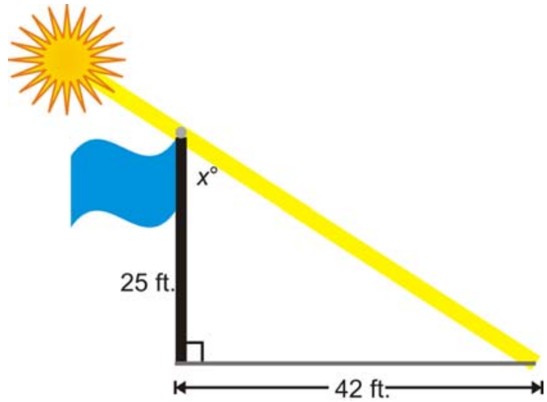
$$\tan 87.4^\circ = \frac{h}{25}$$
$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

According to Wikipedia, the actual height of the monument is 555.427 ft.

Example B

A 25 foot tall flagpole casts a 42 foot shadow. What is the angle that the sun hits the flagpole?



Draw a picture. The angle that the sun hits the flagpole is x°. We need to use the inverse tangent ratio.

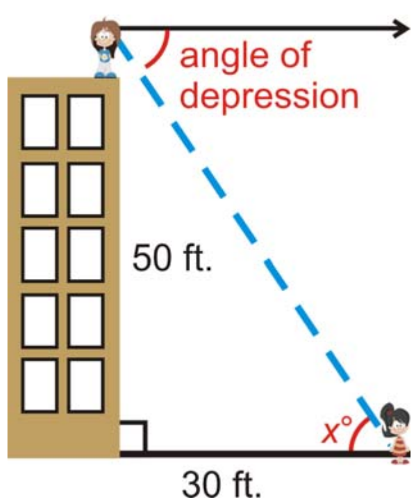
$$\tan x = \frac{42}{25}$$
$$\tan^{-1}\frac{42}{25} \approx 59.2^\circ = x$$

Example C

Elise is standing on top of a 50 foot building and sees her friend, Molly. If Molly is 30 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.

Because of parallel lines, the angle of depression is equal to the angle at Molly, or x°. We can use the inverse tangent ratio.

$$\tan^{-1}\left(\frac{54.5}{30}\right) = 61.2^\circ = x$$



Watch this video for help with the Examples above.

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52514

CK-12 Foundation: Chapter8TrigonometryWordProblemsB

Concept Problem Revisited

To find the horizontal length and the actual length of the ramp, we need to use the tangent and sine.

$$\tan 5^\circ = \frac{2}{x} \qquad\qquad \sin 5^\circ = \frac{2}{y}$$

$$x = \frac{2}{\tan 5^\circ} = 22.86 \qquad\qquad y = \frac{2}{\sin 5^\circ} = 22.95$$

Vocabulary

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used. The three trigonometric (or trig) ratios are ***sine***, ***cosine***, and ***tangent***. The ***angle of depression*** is the angle measured down from the horizon or a horizontal line. The ***angle of elevation*** is the angle measured up from the horizon or a horizontal line.

Guided Practice

1. Mark is flying a kite and realizes that 300 feet of string are out. The angle of the string with the ground is 42.5°. How high is Mark's kite above the ground?

2. A 20 foot ladder rests against a wall. The base of the ladder is 7 feet from the wall. What angle does the ladder make with the ground?

3. A 20 foot ladder rests against a wall. The ladder makes a 55° angle with the ground. How far from the wall is the base of the ladder?

Answers

1. It might help to draw a picture. Then write and solve a trig equation.

$$\begin{aligned} \sin 42.5^\circ &= \frac{x}{300} \\ 300 \cdot \sin 42.5^\circ &= x \\ x &\approx 202.7 \end{aligned}$$

The kite is about 202.7 feet off of the ground.

2. It might help to draw a picture.

$$\cos x = \frac{7}{20}$$
$$x = \cos^{-1}\frac{7}{20}$$
$$x \approx 69.5^\circ$$

.

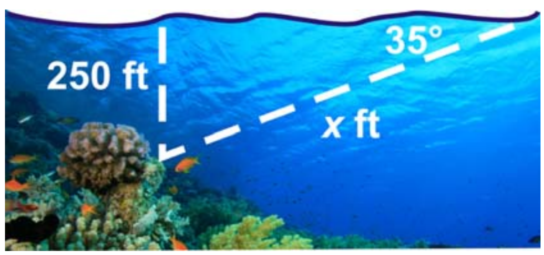
3. It might help to draw a picture.

$$\cos 55^\circ = \frac{x}{20}$$
$$20 \cdot \cos 55^\circ = x$$
$$x \approx 11.5 ft$$

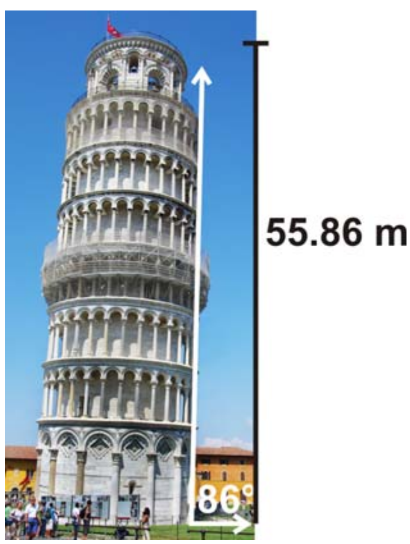
.

Explore More

1. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point is 250 feet. How far away is she from the reef?

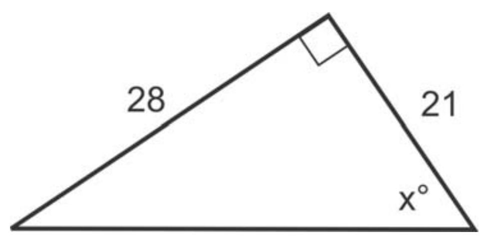


2. The Leaning Tower of Pisa currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?



3. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?

4. William spots a tree directly across the river from where he is standing. He then walks 20 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 65°. How wide is the river?
5. Diego is flying his kite one afternoon and notices that he has let out the entire 120 ft of string. The angle his string makes with the ground is 52°. How high is his kite at this time?
6. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?
7. Upon descent an airplane is 20,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
8. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
9. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
10. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?
11. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree's shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.
12. Alayna is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 5 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
13. Tommy was solving the triangle below and made a mistake. What did he do wrong?



$$\tan^{-1}\left(\frac{21}{28}\right) \approx 36.9^\circ$$

14. Tommy then continued the problem and set up the equation: $\cos 36.9^\circ = \frac{21}{h}$. By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?
15. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

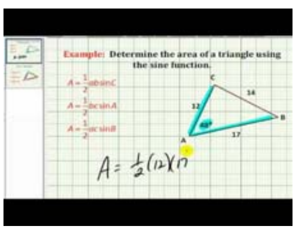
Concept 8 SLT 25 Derive the trigonometric formula for the area of a triangle.

While in the lunch room with your friends one day, you're discussing different ways you can use the things you've learned in math class. You tell your friends that you've been learning a lot about triangles, such as how to find their area. One of your friends looks down at your plate and starts to smile.

"Alright," he says. "If you're so good at things involving triangles, I dare you to find something simple. Tell me the area of your slice of pizza." He points down at the pizza on your plate.

The pizza is shaped like a triangle. But unfortunately its not a right triangle. The outer edge is 5 inches long, and the long sides are 7 inches long. The angle between the edge and the long side of the slice is 69°. Is there any way to tell the area of your pizza slice?

Watch This



MEDIA

Click image to the left for use the URL below.

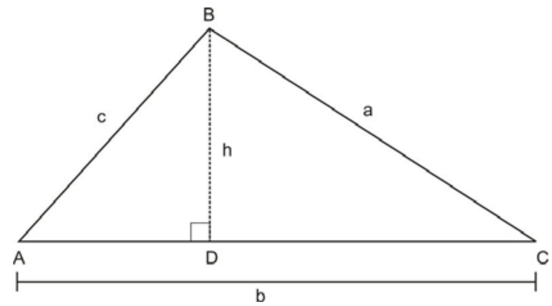
URL: http://www.ck12.org/flx/render/embeddedobject/53108

James Sousa Example: Determine the Area of a Triangle Using the Sine Function

Guidance

We can use the area formula from Geometry, $A = \frac{1}{2}bh$, as well as the sine function, to derive a new formula that can be used when the height, or altitude, of a triangle is unknown.

In $\triangle ABC$ below, BD is altitude from B to AC. We will refer to the length of BD as h since it also represents the height of the triangle. Also, we will refer to the area of the triangle as K to avoid confusing the area with $\angle A$.



$k = \frac{1}{2}bh$	Area of a triangle
$k = \frac{1}{2}b(c\sin A)$	$\sin A = \frac{h}{c}$ therefore $c\sin A = h$
$k = \frac{1}{2}bc\sin A$	Simplify

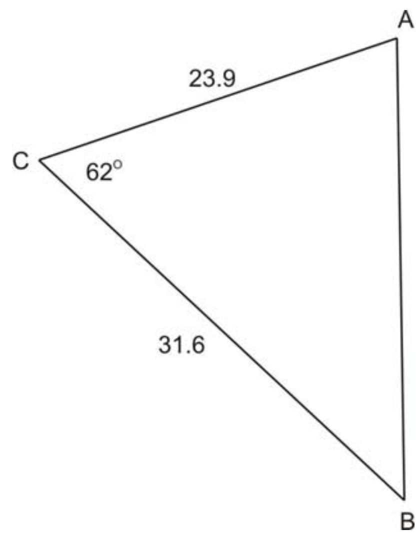
We can use a similar method to derive all three forms of the area formula, regardless of the angle:

$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}ac\sin B$$
$$K = \frac{1}{2}ab\sin C$$

The formula $K = \frac{1}{2}\,bc\sin A$ requires us to know two sides and the included angle (SAS) in a triangle. Once we know these three things, we can easily calculate the area of an oblique triangle.

Example A

In $\triangle ABC, \angle C = 62^\circ, b = 23.9$, and $a = 31.6$. Find the area of the triangle.

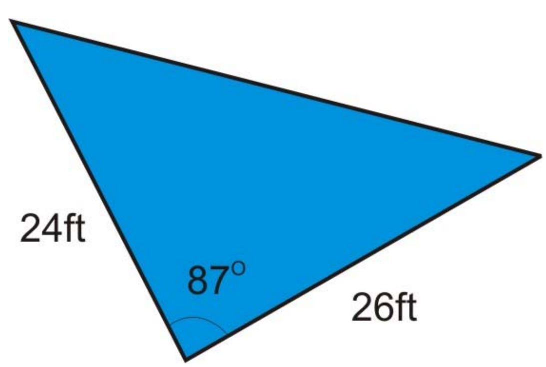


Solution: Using our new formula, $K = \frac{1}{2}ab\sin C$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(31.6)(23.9)\sin 62$$
$$K \approx 333.4$$

Example B

The Pyramid Hotel recently installed a triangular pool. One side of the pool is 24 feet, another side is 26 feet, and the angle in between the two sides is 87°. If the hotel manager needs to order a cover for the pool, and the cost is $35 per square foot, how much can he expect to spend?



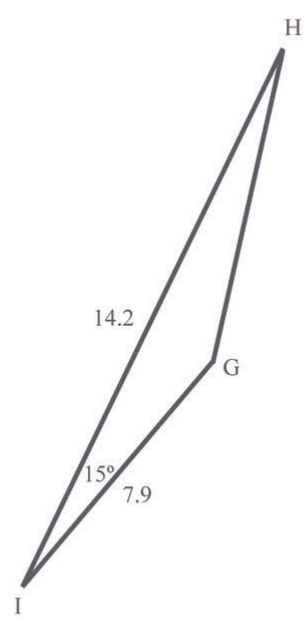
Solution: In order to find the cost of the cover, we first need to know the area of the cover. Once we know how many square feet the cover is, we can calculate the cost. In the illustration above, you can see that we know two of the sides and the included angle. This means we can use the formula $K = \frac{1}{2}\, bc \sin A$.

$$K = \frac{1}{2}\,(24)(26)\sin 87$$
$$K \approx 311.6$$
$$311.6\; sq.ft. \times \$35/sq.ft. = \$10,905.03$$

The cost of the cover will be $\$10,905.03$.

Example C

In $\triangle GHI, \angle I = 15^\circ, g = 14.2$, and $h = 7.9$. Find the area of the triangle.



Solution: Using our new formula, $K = \frac{1}{2}\, ab \sin C$, which is the same as $K = \frac{1}{2}\, gh \sin I$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(14.2)(7.9)\sin 15$$
$$K \approx 14.52$$

Vocabulary

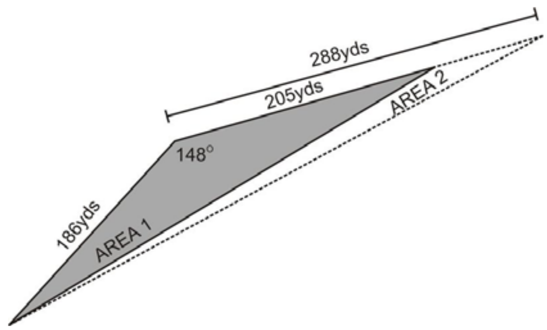
Oblique Triangle: An ***oblique triangle*** is a triangle that does not have 90° as one of its interior angles.

SAS Triangle: An ***SAS triangle*** is a triangle where two sides and the angle in between them are known quantities.

Guided Practice

1. A farmer needs to replant a triangular section of crops that died unexpectedly. One side of the triangle measures 186 yards, another measures 205 yards, and the angle formed by these two sides is 148°.

What is the area of the section of crops that needs to be replanted?



2. The farmer goes out a few days later to discover that more crops have died. The side that used to measure 205 yards now measures 288 yards. How much has the area that needs to be replanted increased by?

3. Find the perimeter of the quadrilateral at the left If the area of $\triangle DEG = 56.5$ and the area of $\triangle EGF = 84.7$.

Solutions:

1. Use $K = \frac{1}{2}\, bc \sin A, K = \frac{1}{2}(186)(205)\sin 148^\circ$. So, the area that needs to be replaced is 10102.9 square yards.

2. $K = \frac{1}{2}(186)(288)\sin 148^\circ = 14193.4$, the area has increased by 4090.5 yards.

3. You need to use the $K = \frac{1}{2}\, bc \sin A$ formula to find DE and GF.

$$56.5 = \frac{1}{2}(13.6)DE \sin 39^\circ \rightarrow DE = 13.2 \qquad 84.7 = \frac{1}{2}(13.6)EF \sin 60^\circ \rightarrow EF = 14.4$$

Second, you need to find sides DG and GF using the Law of Cosines.

$$DG^2 = 13.2^2 + 13.6^2 - 2 \cdot 13.2 \cdot 13.6 \cdot \cos 39^\circ \rightarrow DG = 8.95$$
$$GF^2 = 14.4^2 + 13.6^2 - 2 \cdot 14.4 \cdot 13.6 \cdot \cos 60^\circ \rightarrow GF = 14.0$$

The perimeter of the quadrilateral is 50.55.

Concept Problem Solution

$K = \frac{1}{2}\, bc \sin A$

where in this case, one of the sides is equal to 5, the other is equal to 7, and the angle is 69°.

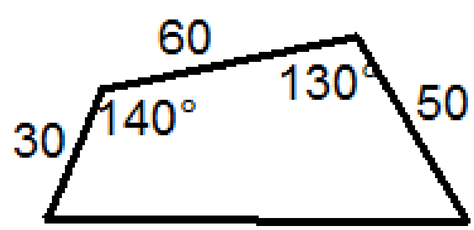
$K = \frac{1}{2}(5)(7)\sin 69^\circ = 16.34 in^2$

Practice

Find the area of each triangle.

1. $\triangle ABC$ if a=13, b=15, and $m\angle C = 71^\circ$.
2. $\triangle ABC$ if b=8, c=4, and $m\angle A = 67^\circ$.
3. $\triangle ABC$ if b=34, c=29, and $m\angle A = 138^\circ$.
4. $\triangle ABC$ if a=3, b=7, and $m\angle C = 80^\circ$.
5. $\triangle ABC$ if a=4.8, c=3.7, and $m\angle B = 43^\circ$.
6. $\triangle ABC$ if a=12, b=5, and $m\angle C = 20^\circ$.
7. $\triangle ABC$ if a=3, b=10, and $m\angle C = 50^\circ$.
8. $\triangle ABC$ if a=5, b=9, and $m\angle C = 14^\circ$.
9. $\triangle ABC$ if a=5, b=7, and c=11.
10. $\triangle ABC$ if a=7, b=8, and c=9.
11. $\triangle ABC$ if a=12, b=14, and c=4.
12. A farmer measures the three sides of a triangular field and gets 114, 165, and 257 feet. What is the measure of the largest angle of the triangle?
13. Using the information from the previous problem, what is the area of the field?

Another field is a quadrilateral where three sides measure 30, 50, and 60 yards, and two angles measure 130° and 140°, as shown below.



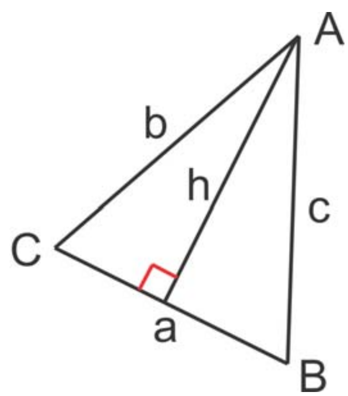
14. Find the area of the quadrilateral. Hint: divide the quadrilateral into two triangles and find the area of each.
15. Find the length of the fourth side.
16. Find the measures of the other two angles.

CONCEPT 9 SLT 26 Use the Law of Sines to solve problems.

A triangle has two angles that measure 60° and 45°. The length of the side between these two angles is 10. What are the lengths of the other two sides?

Guidance

Consider the non right triangle below. We can construct an altitude from any one of the vertices to divide the triangle into two right triangles as show below.



from the diagram we can write two trigonometric functions involving h:

$$\sin C = \frac{h}{b} \qquad and \qquad \sin B = \frac{h}{c}$$
$$b\sin C = h \qquad\qquad c\sin B = h$$

Since both are equal to h, we can set them equal to each other to get:

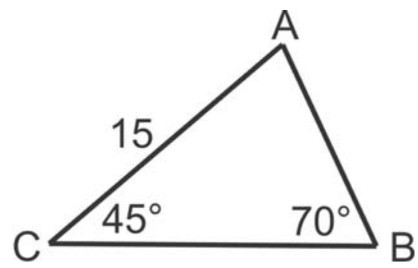
$b\sin C = c\sin B$ and finally divide both sides by bc to create the proportion:

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

If we construct the altitude from a different vertex, say B, we would get the proportion: $\frac{\sin A}{a} = \frac{\sin C}{c}$. Now, the transitive property allows us to conclude that $\frac{\sin A}{a} = \frac{\sin B}{b}$. We can put them all together as the Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. In the examples that follow we will use the Law of Sines to solve triangles.

Example A

Solve the triangle.



Solution: Since we are given two of the three angles in the triangle, we can find the third angle using the fact that the three angles must add up to 180°. So, $m\angle A = 180^\circ - 45^\circ - 70^\circ = 650^\circ$. Now we can substitute the known values into the Law of Sines proportion as shown:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$

Taking two ratios at a time we can solve the proportions to find a and c using cross multiplication.

To find a:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15}$$
$$a = \frac{15 \sin 65^\circ}{\sin 70^\circ} \approx 14.5$$

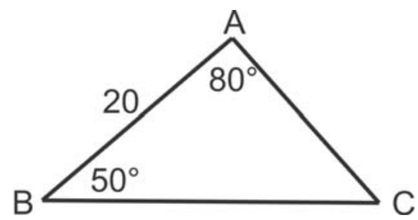
To find c:

$$\frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$
$$c = \frac{15 \sin 45^\circ}{\sin 70^\circ} \approx 11.3$$

This particular triangle is an example in which we are given two angles and the non-included side or AAS (also SAA).

Example B

Solve the triangle.



Solution: In this example we are given two angles and a side as well but the side is between the angles. We refer to this arrangement as ASA. In practice, in doesn't really matter whether we are given AAS or ASA. We will follow the same procedure as Example A. First, find the third angle: $m\angle A = 180^\circ - 50^\circ - 80^\circ = 50^\circ$.

Second, write out the appropriate proportions to solve for the unknown sides, a and b.

To find a:

$$\frac{\sin 80^\circ}{a} = \frac{\sin 50^\circ}{20}$$
$$a = \frac{20 \sin 80^\circ}{\sin 50^\circ} \approx 25.7$$

To find b:

$$\frac{\sin 50^\circ}{b} = \frac{\sin 50^\circ}{20}$$
$$b = \frac{20 \sin 50^\circ}{\sin 50^\circ} = 20$$

Notice that $c = b$ and $m\angle C = m\angle B$. This illustrates a property of isosceles triangles that states that the base angles (the angles opposite the congruent sides) are also congruent.

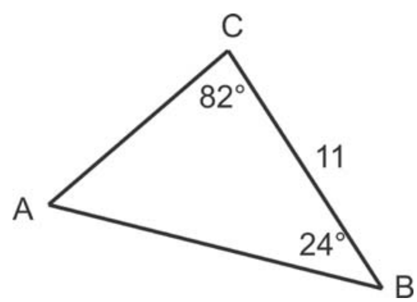
Concept Problem Revisit The measure of the triangle's third angle is $180^\circ - 60^\circ - 45^\circ = 75^\circ$

$$\frac{\sin 45^\circ}{x} = \frac{\sin 75^\circ}{10}, \text{ so } x = \frac{10 \sin 45^\circ}{\sin 75^\circ} \approx 7.29$$
$$\frac{\sin 60^\circ}{y} = \frac{\sin 75^\circ}{10}, \text{ so } y = \frac{10 \sin 60^\circ}{\sin 75^\circ} \approx 8.93$$

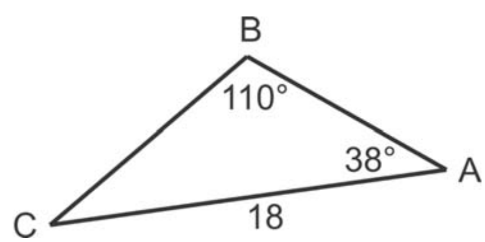
Guided Practice

Solve the triangles.

1.



2.



Answers

1. $m\angle A = 180^\circ - 82^\circ - 24^\circ = 74^\circ$

$$\frac{\sin 24^\circ}{b} = \frac{\sin 74^\circ}{11}, \text{ so } b = \frac{11 \sin 24^\circ}{\sin 74^\circ} \approx 4.7$$
$$\frac{\sin 82^\circ}{c} = \frac{\sin 74^\circ}{11}, \text{ so } c = \frac{11 \sin 82^\circ}{\sin 74^\circ} \approx 11.3$$

2. $m\angle C = 180^\circ - 110^\circ - 38^\circ = 32^\circ$

$$\frac{\sin 38^\circ}{a} = \frac{\sin 110^\circ}{18}, \text{ so } a = \frac{18 \sin 38^\circ}{\sin 110^\circ} \approx 11.8$$
$$\frac{\sin 32^\circ}{c} = \frac{\sin 110^\circ}{18}, \text{ so } c = \frac{18 \sin 32^\circ}{\sin 110^\circ} \approx 10.2$$

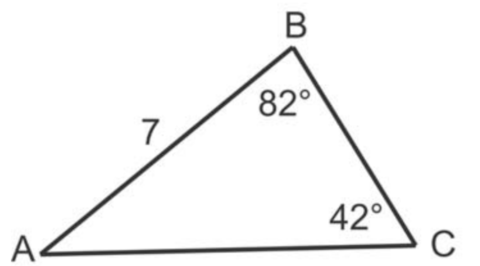
Vocabulary

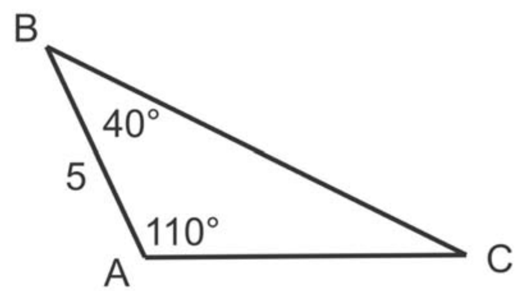
Law of Sines

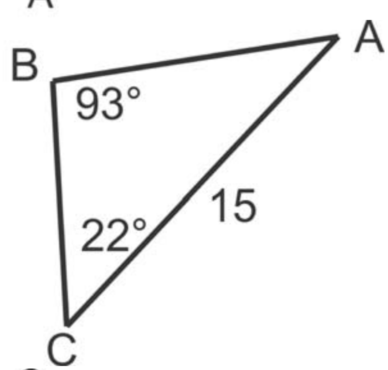
For any triangle, the ratio of the sine of an angle over its opposite side is equal to the sine of any other angle in the triangle over its opposite side. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

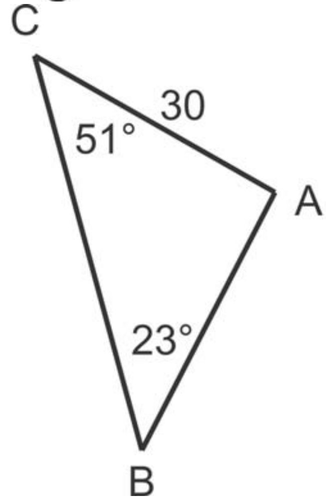
Explore More

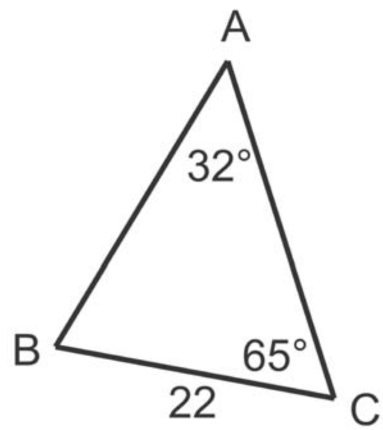
Solve the triangles. Round your answers to the nearest tenth.

1. 

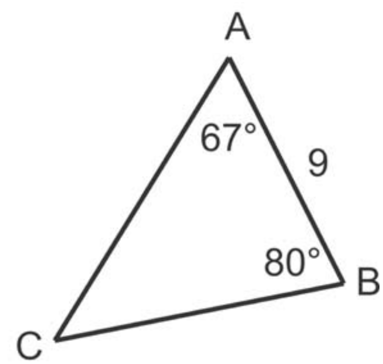
2. 

3. 

4. 



5.



6.

Using the given information, solve ΔABC.

7.

$$m\angle A = 85^\circ$$
$$m\angle C = 40^\circ$$
$$a = 12$$

8.

$$m\angle B = 60^\circ$$
$$m\angle C = 25^\circ$$
$$a = 28$$

9.

$$m\angle B = 42^\circ$$
$$m\angle A = 36^\circ$$
$$b = 8$$

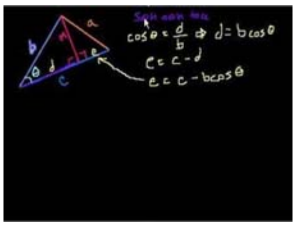
10.

$$m\angle B = 30^\circ$$
$$m\angle A = 125^\circ$$
$$c = 45$$

CONCEPT 10 SLT 27 Use the Law of Cosines to solve problems.

The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

Watch This



MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58142

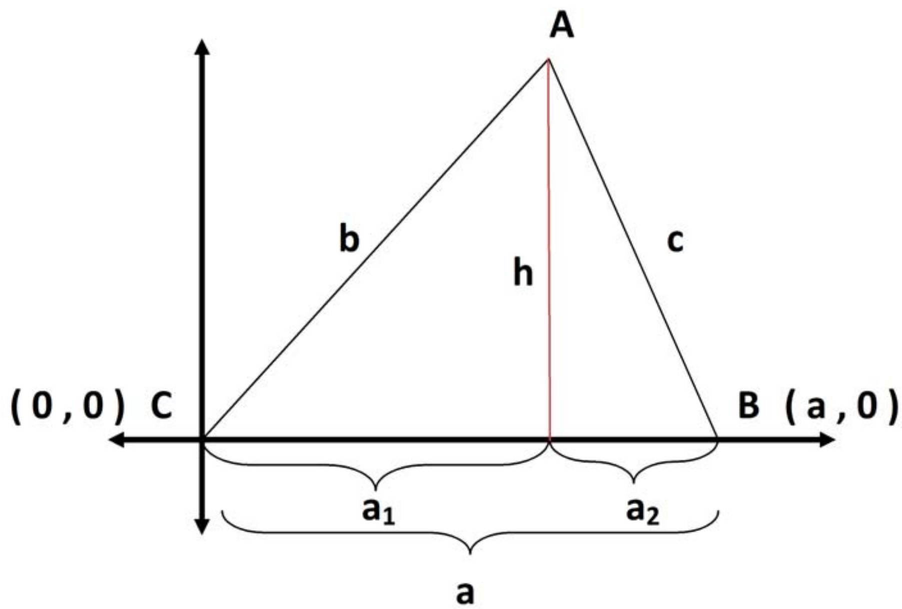
http://www.youtube.com/watch?v=pGaDcOMdw48 Khan Academy: Law of Cosines

Guidance

The Law of Cosines is:

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

It is important to understand the proof:



You know four facts from the picture:

$a = a_1 + a_2 \qquad (1)$

$b^2 = a_1^2 + h^2 \qquad (2)$

$c^2 = a_2^2 + h^2 \qquad (3)$

$\cos C = \frac{a_1}{b} \qquad (4)$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$a_2 = a - a_1 \qquad (5, \text{ from } 1)$

$a_1 = b \cdot \cos C \qquad (6, \text{ from } 4)$

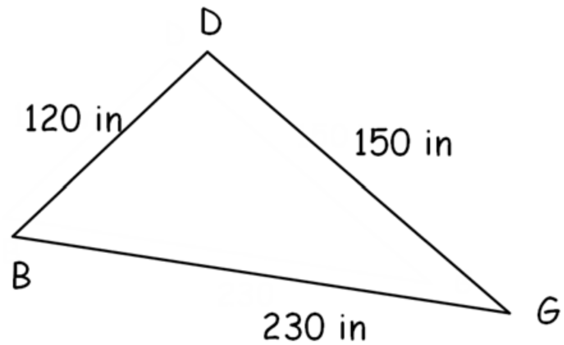
Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

$$\begin{aligned}
c^2 &= a_2^2 + h^2 && \text{(3 again)}\\
c^2 &= (a - a_1)^2 + h^2 && \text{(substitute using 5)}\\
c^2 &= a^2 - 2a \cdot a_1 + a_1^2 + h^2 && \text{(FOIL)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + a_1^2 + h^2 && \text{(substitute using 6)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + b^2 && \text{(substitute using 2)}\\
c^2 &= a^2 + b^2 - 2ab \cdot \cos C && \text{(rearrange terms)}
\end{aligned}$$

There are only two types of problems in which it is appropriate to use the Law of Cosines. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called SSS like in geometry. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called SAS.

Example A

Determine the measure of angle D.

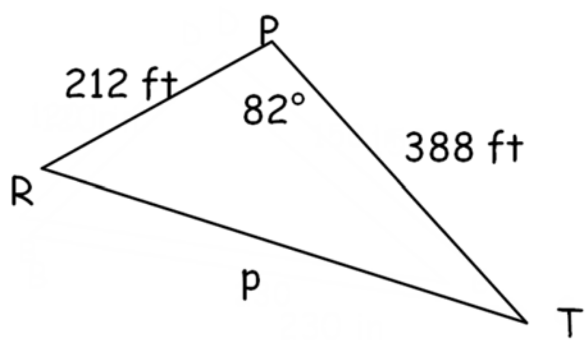


Solution: It is necessary to set up the Law of Cosines equation very carefully with D corresponding to the opposite side of 230. The letters are not ABC like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ 230^2 &= 120^2+150^2-2\cdot 120\cdot 150\cdot\cos D \\ 230^2-120^2-150^2 &= -2\cdot 120\cdot 150\cdot\cos D \\ \frac{230^2-120^2-150^2}{-2\cdot 120\cdot 150} &= \cos D \\ D &= \cos^{-1}\left(\frac{230^2-120^2-150^2}{-2\cdot 120\cdot 150}\right)\approx 116.4^\circ \end{aligned}$$

Example B

Determine the length of side p.

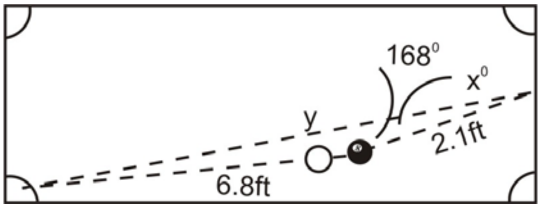


Solution:

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ p^2 &= 212^2+388^2-2\cdot 212\cdot 388\cdot\cos 82^\circ \\ p^2 &\approx 194192.02\ldots \\ p &\approx 440.7 \end{aligned}$$

Example C

In a game of pool, a player must put the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a 168° angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?



Note: This is actually a trick shot performed by spinning the eight ball, and the eight ball will not actually travel in straight-line trajectories. However, to simplify the problem, assume that it travels in straight lines.

Solution: In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X).

$$\begin{aligned}y^2 &= 6.8^2 + 2.1^2 - 2(6.8)(2.1)\cos 168^\circ \\ y^2 &= 78.59 \\ y &= 8.86\ feet\end{aligned}$$

The distance from the spot on the bumper to the pocket is 8.86 feet. We can now use this distance and the Law of Sines to find angle X. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since we know all three sides this problem will yield only one solution.

$$\begin{aligned}\frac{\sin 168^\circ}{8.86} &= \frac{\sin X}{6.8} \\ \frac{6.8 \sin 168^\circ}{8.86} &= \sin X \\ 0.1596 &\approx \sin B \\ \angle B &= 8.77^\circ\end{aligned}$$

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios. We will also look at another real-world application involving the SSA case.

Concept Problem Revisited

A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\ 11^2 &= 12^2 + (13)^2 - 2 \cdot 12 \cdot 13 \cdot \cos C \\ C &= \cos^{-1}\left(\frac{11^2 - 12^2 - 13^2}{-2 \cdot 12 \cdot 13}\right) \approx 52.02\ldots^\circ\end{aligned}$$

Vocabulary

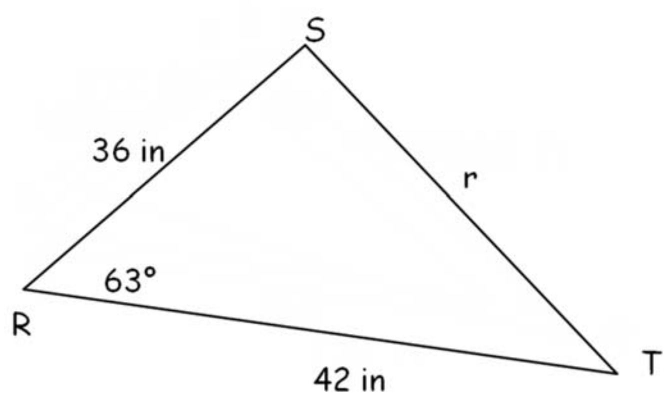
The **Law of Cosines** is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle.

SSS refers to Side, Side, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that all three sides are known in the problem.

SAS refers to Side, Angle, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that the known quantities of a triangle are two sides and the included angle.

Included angle is the angle between two sides.

Guided Practice



1. Determine the length of side r.

2. Determine the measure of angle T.

3. Determine the measure of angle S.

Answers:

1. $r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r = 41.07\ldots$

2. $36^2 = (41.07\ldots)^2 + 42^2 - 2 \cdot (41.07\ldots) \cdot 42 \cdot \cos T$

$T \approx 51.34\ldots^\circ$

3. You could repeat the process from the previous question, or use the knowledge that the three angles in a triangle add up to 180.

$$63 + 51.34\ldots + S = 180$$
$$S \approx 65.65^\circ$$

Practice

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$.

1. Find the measure of angle A.

2. Find the measure of angle B.

3. Find the measure of angle C.

4. Find the measure of angle C in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$.

5. Find the measure of angle D.

6. Find the measure of angle E.

7. Find the measure of angle F.

In $\Delta GHI, g = 19, \angle H = 55^\circ$, and $i = 12$.

8. Find the length of h.

9. Find the measure of angle G.

10. Find the measure of angle I.

11. Explain why the Law of Cosines is connected to the Pythagorean Theorem.

12. What are the two types of problems where you might use the Law of Cosines?

Determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$

14. $a = 1, b = 5, c = 4$

15. $a = 5, b = 6, c = 10$

References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
4. CK-12 Foundation. . CCSA

CONCEPT 11 SLT 28 Apply the Law of Sines and the Law of Cosines to solve problems.

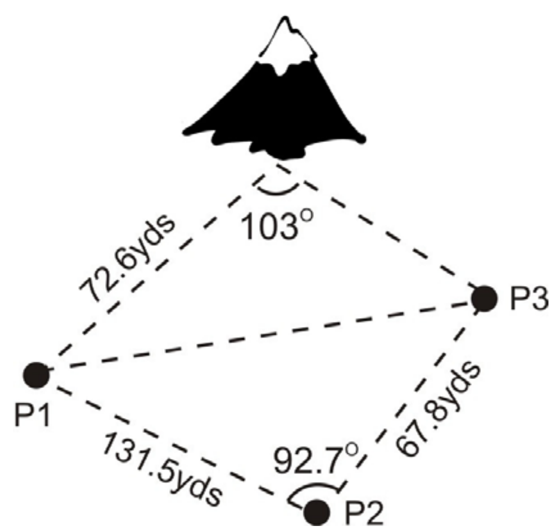
While helping your mom bake one day, the two of you get an unusual idea. You want to cut the cake into pieces, and then frost over the surface of each piece. You start by cutting out a slice of the cake, but you don't quite cut the slice correctly. It ends up being an oblique triangle, with a 5 inch side, a 6 inch side, and an angle of 70° between the sides you measured. Can you help your mom determine the length of the third side, so she can figure out how much frosting to put out?

Guidance

The Law of Cosines and the Law of Sines may be utilized to analyze real world situations.

Example A

Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain. The mountains forms a 103° angle with Person 1 and Person 3, while Person 2 forms a 92.7° angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.



Solution: In the triangle formed by the three people, we know two sides and the included angle (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

To find x:

$$\begin{aligned} x^2 &= 131.5^2 + 67.8^2 - 2(131.5)(67.8)\cos 92.7 \\ x^2 &= 22729.06397 \\ x &= 150.8\ yds \end{aligned}$$

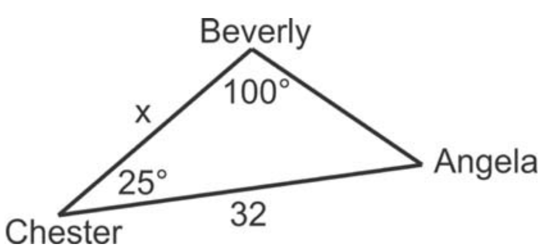
Now that we know $x = 150.8$, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since $150.8 > 72.6$, we know that we will have only one solution to this problem.

$$\begin{aligned} \frac{\sin 103}{150.8} &= \frac{\sin Y}{72.6} \\ \frac{72.6 \sin 103}{150.8} &= \sin Y \\ 0.4690932805 &= \sin Y \\ 28.0 &\approx \angle Y \end{aligned}$$

Example B

Three fishing ships in a fleet are out on the ocean. The Chester is 32 km from the Angela. An officer on the Chester measures the angle between the Angela and the Beverly to be 25°. An officer on the Beverly measures the angle between the Angela and the Chester to be 100°. How far apart, to the nearest kilometer are the Chester and the Beverly?

Solution: First, draw a picture. Keep in mind that when we say that an officer on one of the ships is measuring an angle, the angle she is measuring is at the vertex where her ship is located.



Now that we have a picture, we can determine the angle at the Angela and then use the Law of Sines to find the distance between the Beverly and the Chester.

The angle at the Angela is $180° - 100° - 25° = 55°$.

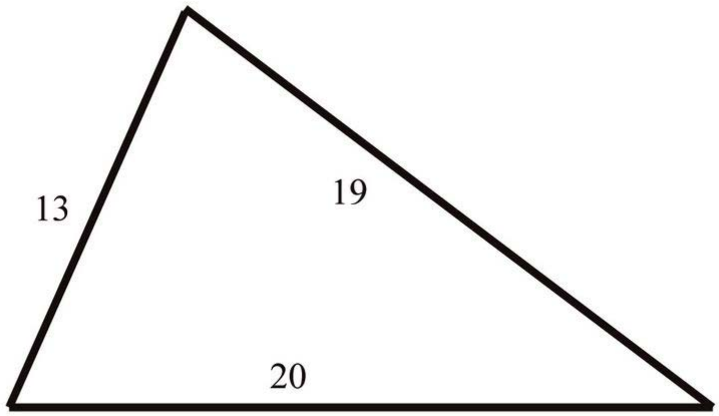
Now find x,

$$\begin{aligned} \frac{\sin 55°}{x} &= \frac{\sin 100°}{32} \\ x &= \frac{32 \sin 55°}{\sin 100°} \approx 27 \end{aligned}$$

The Beverly and the Chester are about 27 km apart.

Example C

Katie is constructing a kite shaped like a triangle.



She knows that the lengths of the sides are a = 13 inches, b = 20 inches, and c = 19 inches. What is the measure of the angle between sides "a" and "b"?

Solution: Since she knows the length of each of the sides of the triangle, she can use the Law of Cosines to find the angle desired:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2(a)(b)\cos C \\ 19^2 &= 13^2 + 20^2 - (2)(13)(20)\cos C \\ 361 &= 169 + 400 - 520\cos C \\ -208 &= -520\cos C \\ \cos C &= 0.4 \\ C &\approx 66.42^\circ \end{aligned}$$

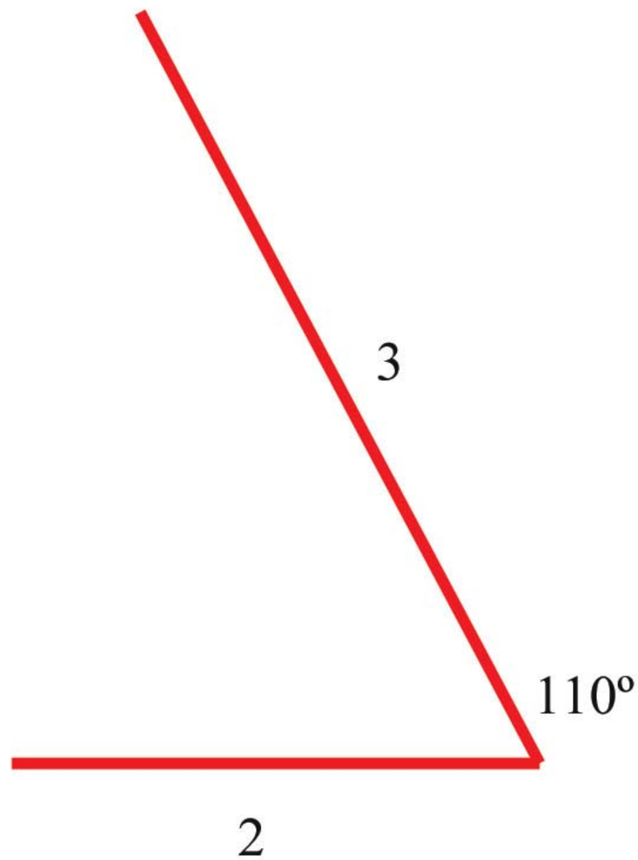
Vocabulary

Law of Cosines: The ***law of cosines*** is a rule involving the sides of an oblique triangle stating that the square of a side of the triangle is equal to the sum of the squares of the other two sides plus two times the lengths of the other two sides times the cosine of the angle opposite the side being computed.

Guided Practice

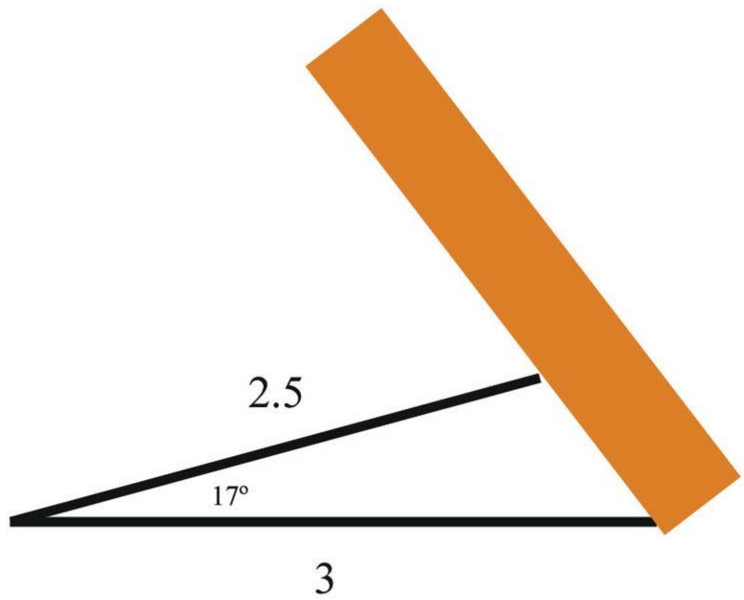
1. A surveying team is measuring the distance between point A on one side of a river and point B on the far side of the river. One surveyor is positioned at point A and the second surveyor is positioned at point C, 65 m up the riverbank from point A. The surveyor at point A measures the angle between points B and C to be 103°. The surveyor at point C measures the angle between points A and B to be 42°. Find the distance between points $A$and B.

2. While hiking one day you walk for 2 miles in one direction. You then turn 110° to the left and walk for 3 more miles. Your path looks like this:



When you turn to the left again to complete the triangle that is your hiking path for the day, how far will you have to walk to complete the third side? What angle should you turn before you start walking back home?

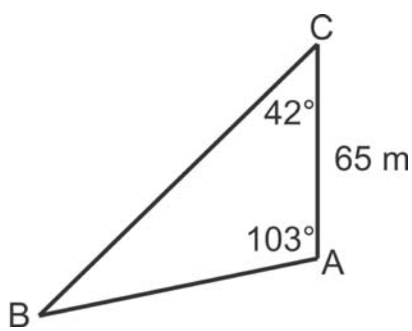
3. A support at a construction site is being used to hold up a board so that it makes a triangle, like this:



If the angle between the support and the ground is 17°, the length of the support is 2.5 meters, and the distance between where the board touches the ground and the bottom of the support is 3 meters, how far along the board is the support touching? What is the angle between the board and the ground?

Solutions:

1.



$$m\angle B = 180^\circ - 103^\circ - 42^\circ = 35^\circ$$
$$\frac{\sin 35^\circ}{65} = \frac{\sin 42^\circ}{c}$$
$$c = \frac{65 \sin 42^\circ}{\sin 35^\circ} \approx 75.8\ m$$

2. Since you know the lengths of two of the legs of the triangle, along with the angle between them, you can use the Law of Cosines to find out how far you'll have to walk along the third leg:

$$\begin{aligned} c^2 &= a^2+b^2+2ab\cos 70^\circ \\ c^2 &= 4+1+(2)(2)(1)(.342) \\ c^2 &= 6.368 \\ c &= \sqrt{6.368} \approx 2.52 \end{aligned}$$

Now you have enough information to solve for the interior angle of the triangle that is supplementary to the angle you need to turn:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \frac{\sin 70^\circ}{2.52} &= \frac{\sin B}{2} \\ \sin B = \frac{2\sin 70^\circ}{2.52} &= \frac{1.879}{2.52} = .746 \\ B = \sin^{-1}(.746) &= 48.25^\circ \end{aligned}$$

The angle 48.25° is the interior angle of the triangle. So you should turn $90^\circ + (90^\circ - 48.25^\circ) = 90^\circ + 41.75^\circ = 131.75^\circ$ to the left before starting home.

3. You should use the Law of Cosines first to solve for the distance from the ground to where the support meets the board:

$$\begin{aligned} c^2 &= a^2+b^2+2ab\cos 17^\circ \\ c^2 &= 6.25+9+(2)(2.5)(3)\cos 17^\circ \\ c^2 &= 6.25+9+(2)(2.5)(3)(.956) \\ c^2 &= 26.722 \\ c &\approx 5.17 \end{aligned}$$

And now you can use the Law of Sines:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \frac{\sin 17^\circ}{5.17} &= \frac{\sin B}{2.5} \\ \sin B = \frac{2.5\sin 17^\circ}{5.17} &= .1414 \\ B = \sin^{-1}(.1414) &= 8.129^\circ \end{aligned}$$

Concept Problem Solution

You can use the Law of Cosines to help your mom find out the length of the third side on the piece of cake:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
c^2 &= 5^2 + 6^2 + (2)(5)(6)\cos 70^\circ \\
c^2 &= 25 + 36 + 60(.342) \\
c^2 &= 81.52 \\
c &\approx 9.03
\end{aligned}
$$

The piece of cake is just a little over 9 inches long.

Practice

1. While hiking one day you walk for 5 miles due east, then turn to the left and walk 3 more miles 30° west of north. At this point you want to return home. How far are you from home if you were to walk in a straight line?

2. A parallelogram has sides of 20 and 31 ft, and an angle of 46°. Find the length of the longer diagonal of the parallelogram.

3. A surveyor is trying to find the distance across a ravine. He measures the angle between a spot on the far side of the ravine, X, and a spot 200 ft away on his side of the ravine, Y, to be 100°. He then walks to Y the angle between X and his previous location to be 20°. How wide is the ravine?

4. Dirk wants to find the length of a long building from one side (point A) to the other (point B). He stands outside of the building (at point C), where he is 500 ft from point A and 220 ft from point B. The angle at C is 94°. Find the length of the building.

5. A triangular plot of land has angles 46° and 58°. The side opposite the 46° angle is 35 m long. How much fencing, to the nearest half meter, is required to enclose the entire plot of land?

Determine whether or not each triangle is possible.

6. a=12, b=15, c=10

7. a=1, b=5, c=4

8. $\angle A = 32^\circ$, a=8, b=10