CK-12

flexbook
next generation textbooks

CK-12 FlexBook

MCPS C2.0 Geometry Unit 2 Topic 1 FlexBook

MCPS Math
MCPS

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

AUTHORS
MCPS Math
MCPS

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-content, web-based collaborative model termed the **FlexBook®**, CK-12 intends to pioneer the generation and distribution of high-quality educational content that will serve both as core text as well as provide an adaptive environment for learning, powered through the **FlexBook Platform®**.

Printed: October 27, 2014

Contents

Concept 1 SLTs 1 & 2 Draw a dilation when given a rule and write a rule given a dilation.

What if you enlarged or reduced a triangle without changing its shape? How could you find the scale factor by which the triangle was stretched or shrunk? After completing this Concept, you'll be able to use the corresponding sides of dilated figures to solve problems like this one.

Watch This

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52562

CK-12 Foundation: Chapter7DilationA

Learn more about dilations by watching the video at this link.

Guidance

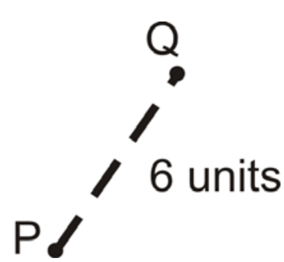
A **transformation** is an operation that moves, flips, or changes a figure to create a new figure. Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.*** A **dilation** makes a figure larger or smaller, but has the same shape as the original. In other words, the dilation is similar to the original. All dilations have a **center** and a **scale factor.** The center is the point of reference for the dilation (like the vanishing point in a perspective drawing) and scale factor tells us how much the figure stretches or shrinks. A scale factor is typically labeled k and is always greater than zero. Also, if the original figure is labeled $\triangle ABC$, for example, the dilation would be $\triangle A'B'C'$. The ' indicates that it is a copy. This tic mark is said "prime," so A' is read "A prime." A second dilation would be A'', read "A double-prime."

If the dilated image is smaller than the original, then the scale factor is $0 < 1$**.**

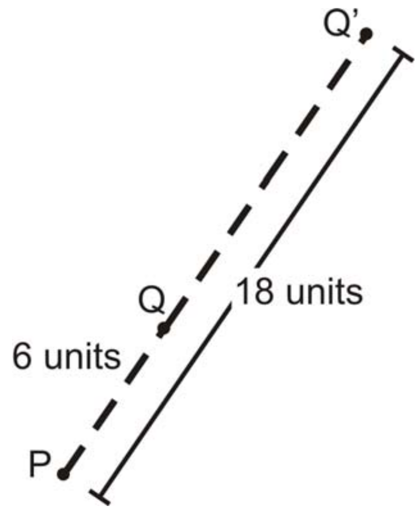
If the dilated image is larger than the original, then the scale factor is $k > 1$**.**

Example A

The center of dilation is P and the scale factor is 3. Find Q'.

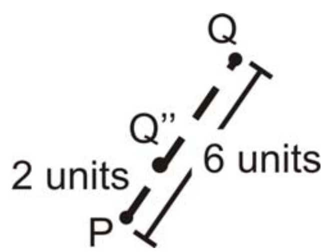


If the scale factor is 3 and Q is 6 units away from P, then Q' is going to be $6 \times 3 = 18$ units away from P. Because we are only dilating apoint, the dilation will be collinear with the original and center.



Example B

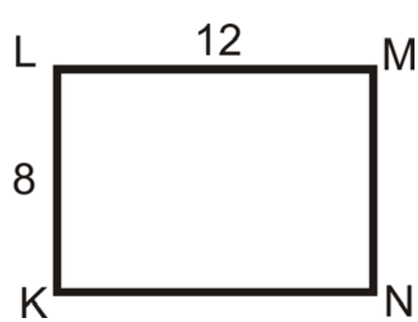
Using the picture above, change the scale factor to $\frac{1}{3}$. Find Q''.



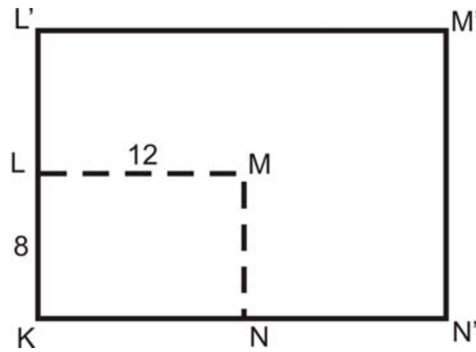
Now the scale factor is $\frac{1}{3}$, so Q'' is going to be $\frac{1}{3}$ the distance away from P as Q is. In other words, Q'' is going to be $6 \times \frac{1}{3} = 2$ units away from P. Q'' will also be collinear with Q and center.

Example C

$KLMN$ is a rectangle with length 12 and width 8. If the center of dilation is K with a scale factor of 2, draw $K'L'M'N'$.



If K is the center of dilation, then K and K' will be the same point. From there, L' will be 8 units above L and N' will be 12 units to the right of N.



Watch this video for help with the Examples above.

MEDIA

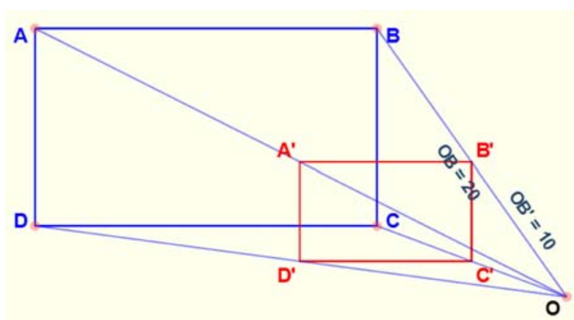
Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52563

CK-12 Foundation: Chapter7DilationB

Example D

ABCD is a rectangle. The center of dilation is *O* and the scale factor is $\frac{1}{2}$. *OB* is 20 units. OB' is half the length of *OB*, which is 10 units.



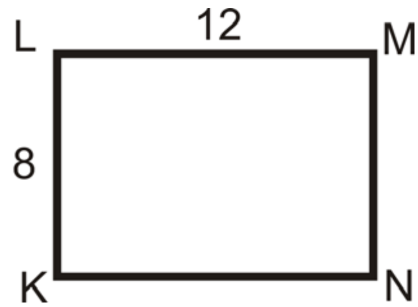
http://www.mathopenref.com/dilate.html

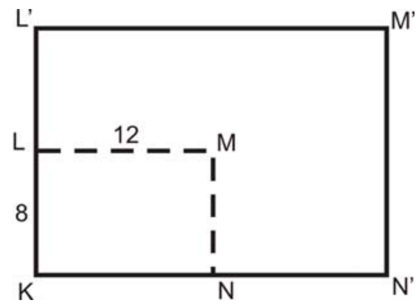
Vocabulary

A ***dilation*** an enlargement or reduction of a figure that preserves shape but not size. All dilations are similar to the original figure. ***Similar*** figures are the same shape but not necessarily the same size. The ***center of a dilation*** is the point of reference for the dilation and the ***scale factor*** for a dilation tells us how much the figure stretches or shrinks.

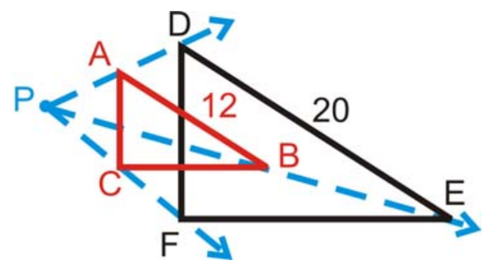
Guided Practice

1. Find the perimeters of *KLMN* and $K'L'M'N'$. Compare this ratio to the scale factor.

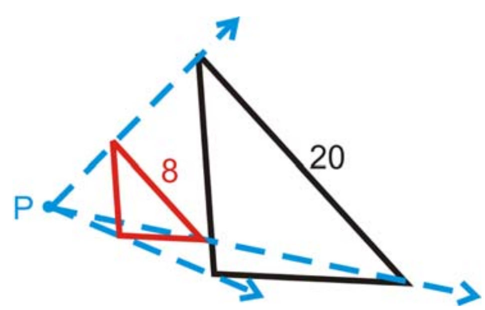




2. $\triangle ABC$ is a dilation of $\triangle DEF$. If P is the center of dilation, what is the scale factor?



3. Find the scale factor, given the corresponding sides. In the diagram, the **black** figure is the original and P is the center of dilation.



Answers:

1. The perimeter of $KLMN = 12+8+12+8 = 40$. The perimeter of $K'L'M'N' = 24+16+24+16 = 80$. The ratio is 80:40, which reduces to 2:1, which is the same as the scale factor.

2. Because $\triangle ABC$ is a dilation of $\triangle DEF$, then $\triangle ABC \sim \triangle DEF$. The scale factor is the ratio of the sides. Since $\triangle ABC$ is smaller than the original, $\triangle DEF$, the scale factor is going to be less than one, $\frac{12}{20} = \frac{3}{5}$.

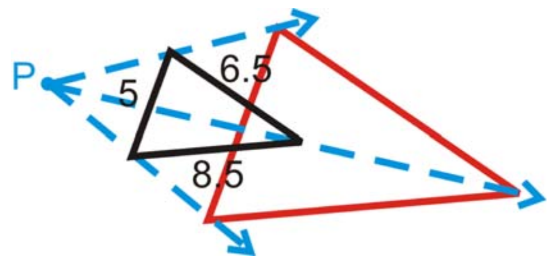
If $\triangle DEF$ was the dilated image, the scale factor would have been $\frac{5}{3}$.

3. Since the dilation is smaller than the original, the scale factor is going to be less than one. $\frac{8}{20} = \frac{2}{5}$

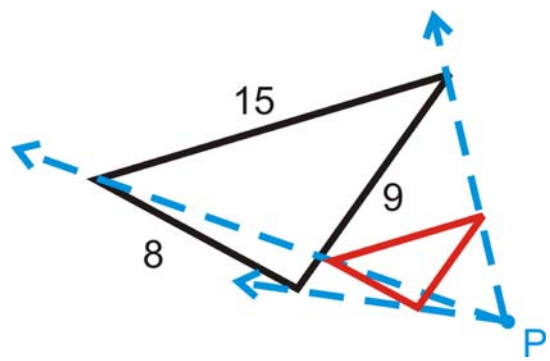
Practice

In the two questions below, you are told the scale factor. Determine the dimensions of the dilation. In each diagram, the **black** figure is the original and P is the center of dilation.

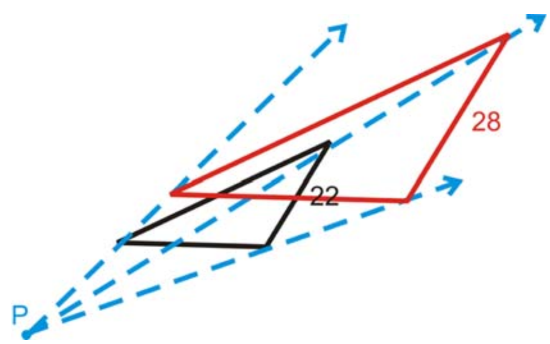
1. $k = 4$



2. $k = \frac{1}{3}$

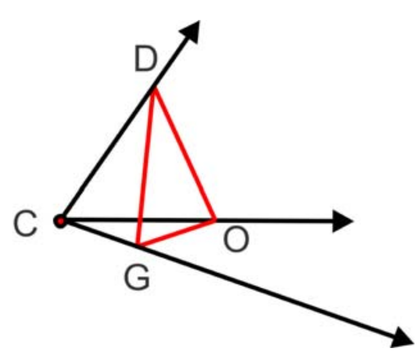


In the question below, find the scale factor, given the corresponding sides. In the diagram, the **black** figure is the original and P is the center of dilation.



3.
4. Find the perimeter of both triangles in #1. What is the ratio of the perimeters?
5. ***Writing*** What happens if $k = 1$?

Construction We can use a compass and straight edge to construct a dilation as well. Copy the diagram below.

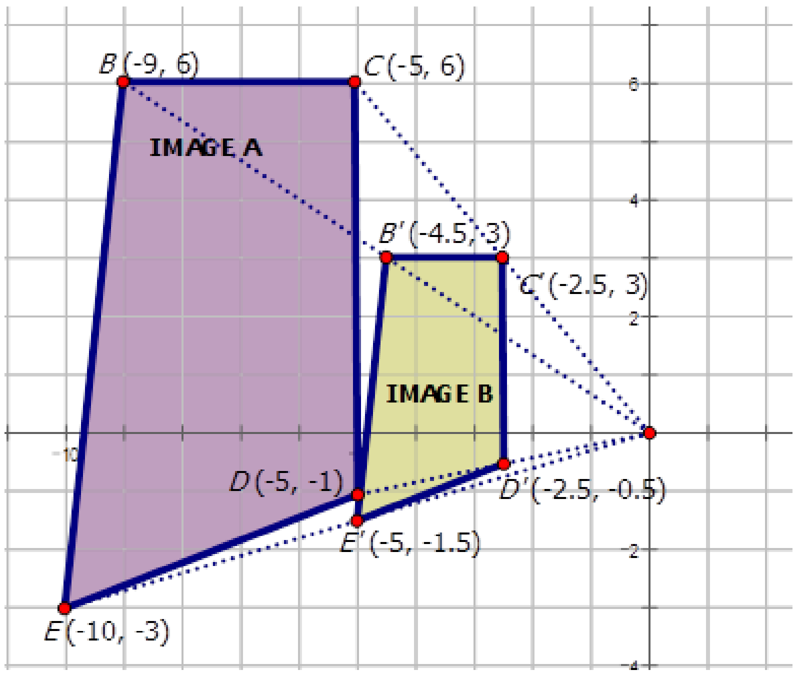


6. Set your compass to be CG and use this setting to mark off a point 3 times as far from C as G is. Label this point G'. Repeat this process for CO and CD to find O' and D'.
7. Connect G', O' and D' to make $\triangle D'O'G'$. Find the ratios, $\frac{D'O'}{DO}, \frac{O'G'}{OG}$ and $\frac{G'D'}{GD}$.
8. What is the scale factor of this dilation?
9. Describe how you would dilate the figure by a scale factor of 4.
10. Describe how you would dilate the figure by a scale factor of $\frac{1}{2}$.

11. The scale factor between two shapes is 1.5. What is the ratio of their perimeters?
12. The scale factor between two shapes is 1.5. What is the ratio of their areas? *Hint: Draw an example and calculate what happens.*
13. Suppose you dilate a triangle with side lengths 3, 7, and 9 by a scale factor of 3. What are the side lengths of the image?
14. Suppose you dilate a rectangle with a width of 10 and a length of 12 by a scale factor of $\frac{1}{2}$. What are the dimensions of the image?
15. Find the areas of the rectangles in #14. What is the ratio of their areas?

Concept 2 SLTs 3 & 4 Draw a dilation when given a rule and write a rule given a dilation. Verify dilations.

The figure below shows a dilation of two trapezoids. Write the mapping rule for the dilation of Image A to Image B.



Watch This

First watch this video to learn about writing rules for dilations.

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/65234

CK-12 Foundation Chapter10RulesforDilationsA

Then watch this video to see some examples.

MEDIA

Click image to the left for use the URL below.

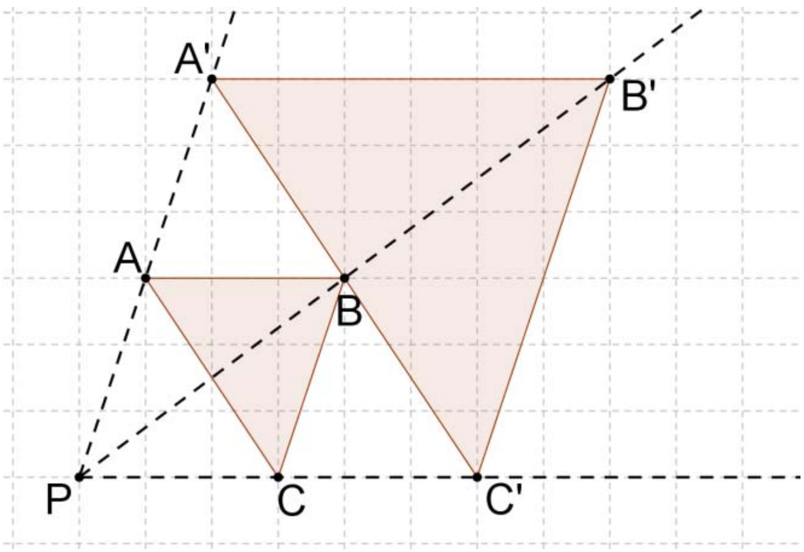
URL: http://www.ck12.org/flx/render/embeddedobject/65235

CK-12 Foundation Chapter10RulesforDilationsB

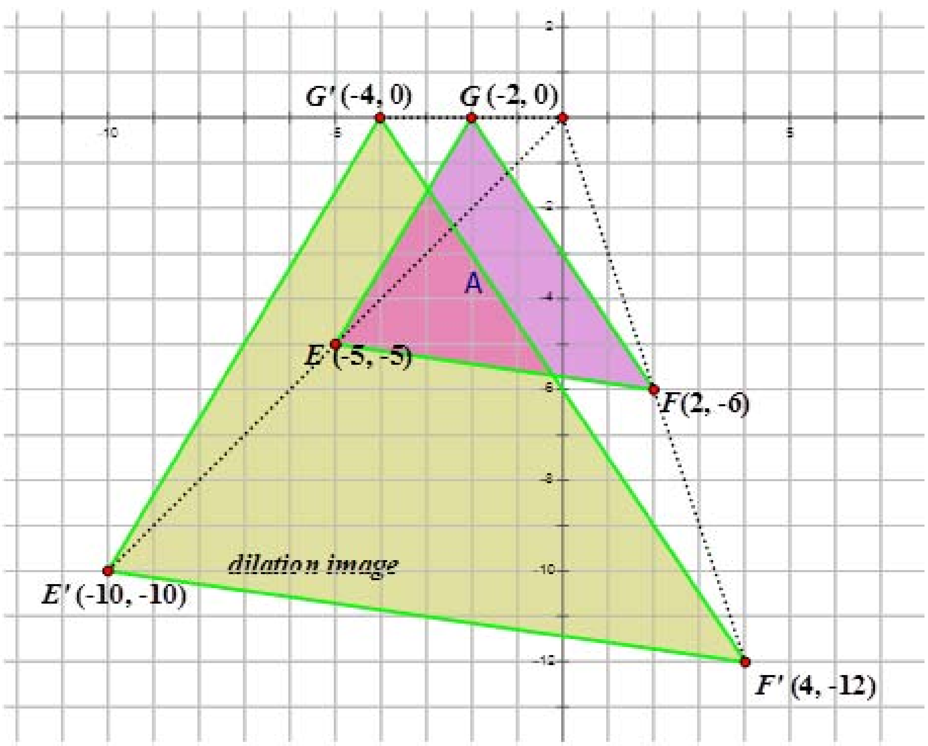
Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage.

ΔABC below has been dilated about point P by a scale factor of 2. Notice that P, A, and A' are all collinear. Similarly, P, B, and B' are collinear and P, C, and C' are collinear. $PC = 3$ and $PC' = 6$. The scale factor of this dilation is 2 because $\frac{PC'}{PC} = \frac{6}{3} = 2$. If you calculate PA, PA', PB and PB' you will find that $\frac{PA'}{PA} = \frac{PB'}{PB} = 2$ as well.



Look at the diagram below:



The Image A has undergone a dilation about the origin with a scale factor of 2. Notice that the points in the dilation image are all double the coordinate points in the preimage. A dilation with a scale factor k about the origin can be described using the following notation:

$$D_k(x,y) = (kx,ky)$$

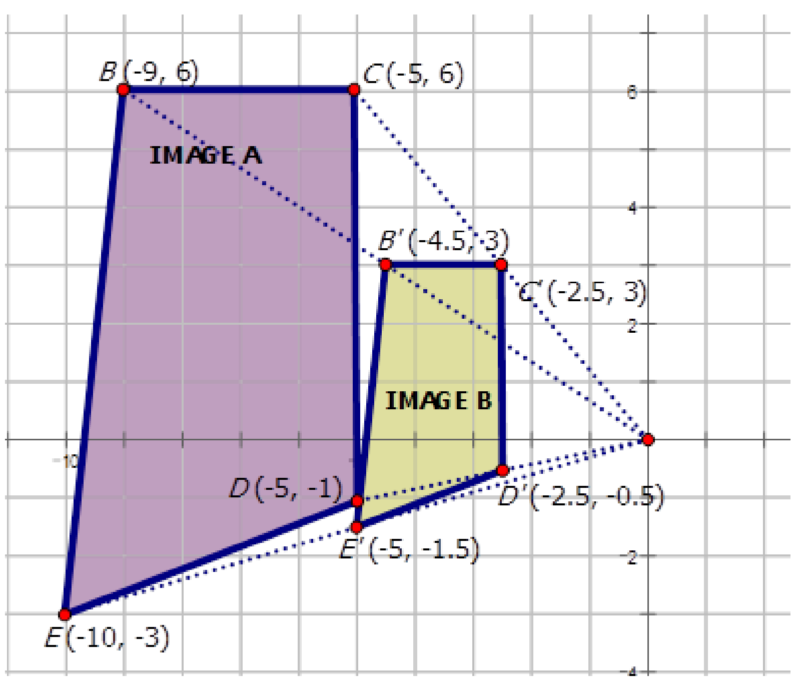
k will always be a value that is greater than 0.

TABLE 2.1:

Scale Factor, k	**Size change for preimage**
$k > 1$	Dilation image is larger than preimage
$0 < k < 1$	Dilation image is smaller than preimage
$k = 1$	Dilation image is the same size as the preimage

Look at the points in each image:

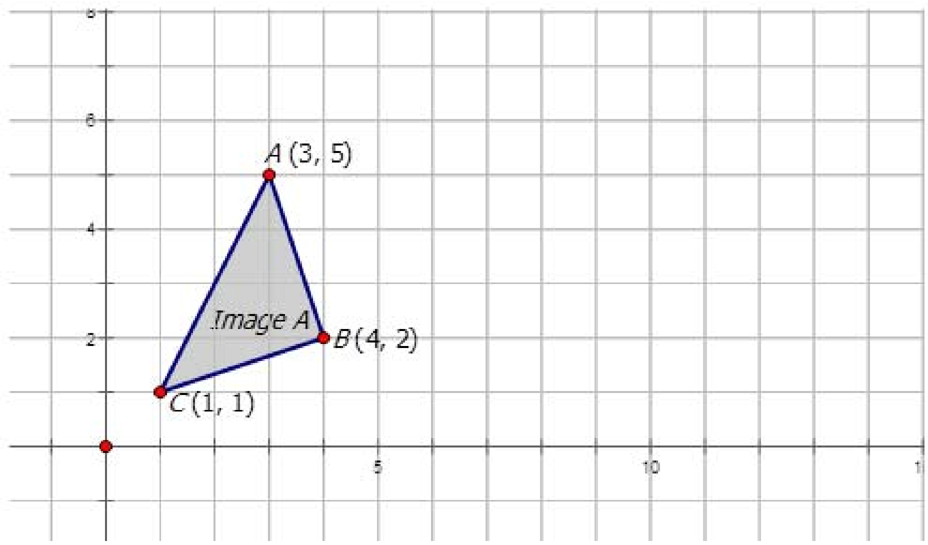
Image A	$B(-9,6)$	$C(-5,6)$	$D(-5,-1)$	$E(-10,-3)$
Image B	$B'(-4.5,3)$	$C'(-2.5,3)$	$D'(-2.5,-0.5)$	$E'(-5,-1.5)$



Notice that the coordinate points in Image B (the dilation image) are $\frac{1}{2}$ that found in Image A. Therefore the Image A undergoes a dilation about the origin of scale factor $\frac{1}{2}$. To write a mapping rule for this dilation you would write: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

Example A

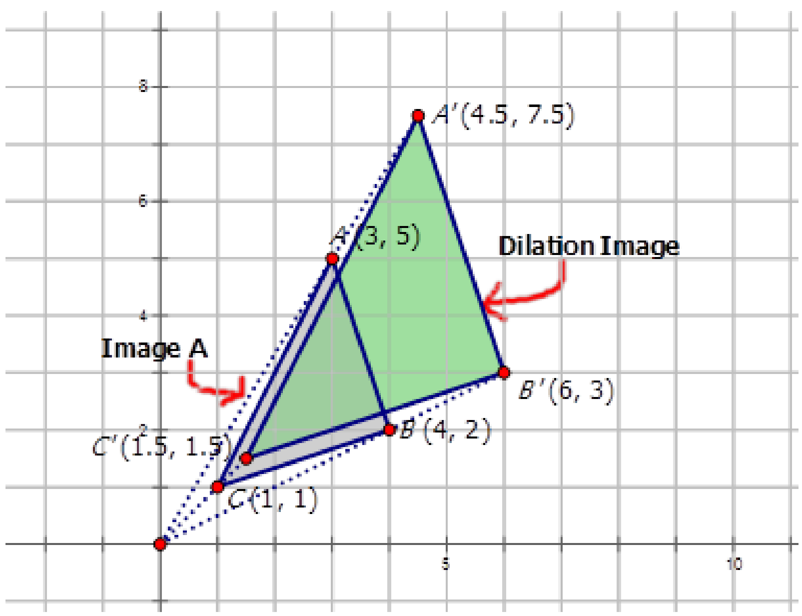
The mapping rule for the dilation applied to the triangle below is $(x,y) \rightarrow (1.5x, 1.5y)$. Draw the dilation image.



Solution: With a scale factor of 1.5, each coordinate point will be multiplied by 1.5.

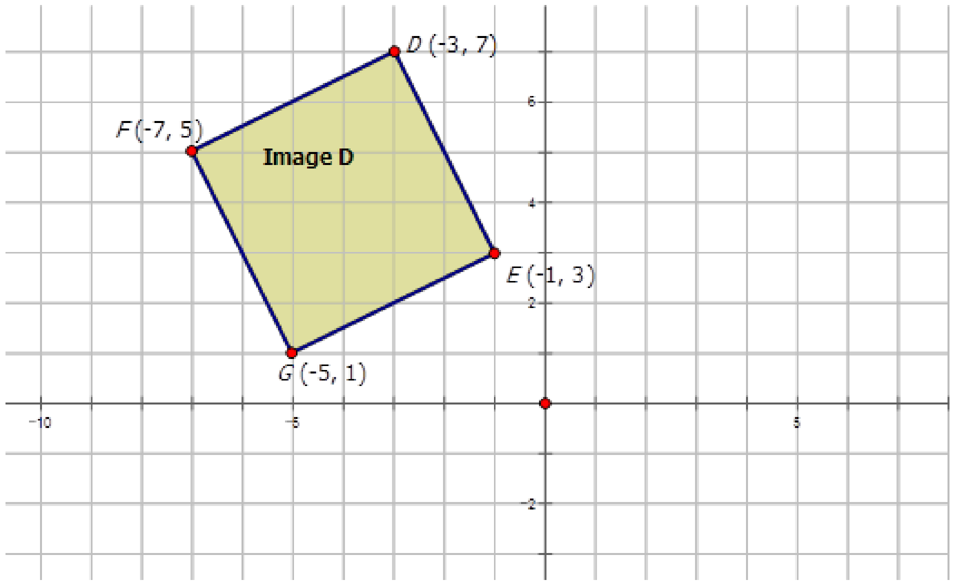
Image A	$A(3,5)$	$B(4,2)$	$C(1,1)$
Dilation Image	$A'(4.5,7.5)$	$B'(6,3)$	$C'(1.5,1.5)$

The dilation image looks like the following:



Example B

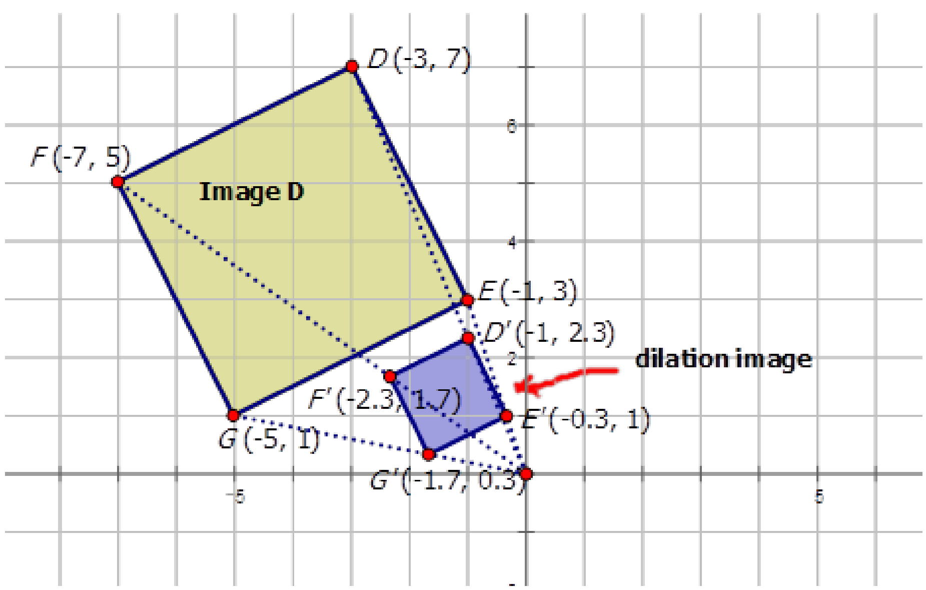
The mapping rule for the dilation applied to the diagram below is $(x,y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$. Draw the dilation image.



Solution: With a scale factor of $\frac{1}{3}$, each coordinate point will be multiplied by $\frac{1}{3}$.

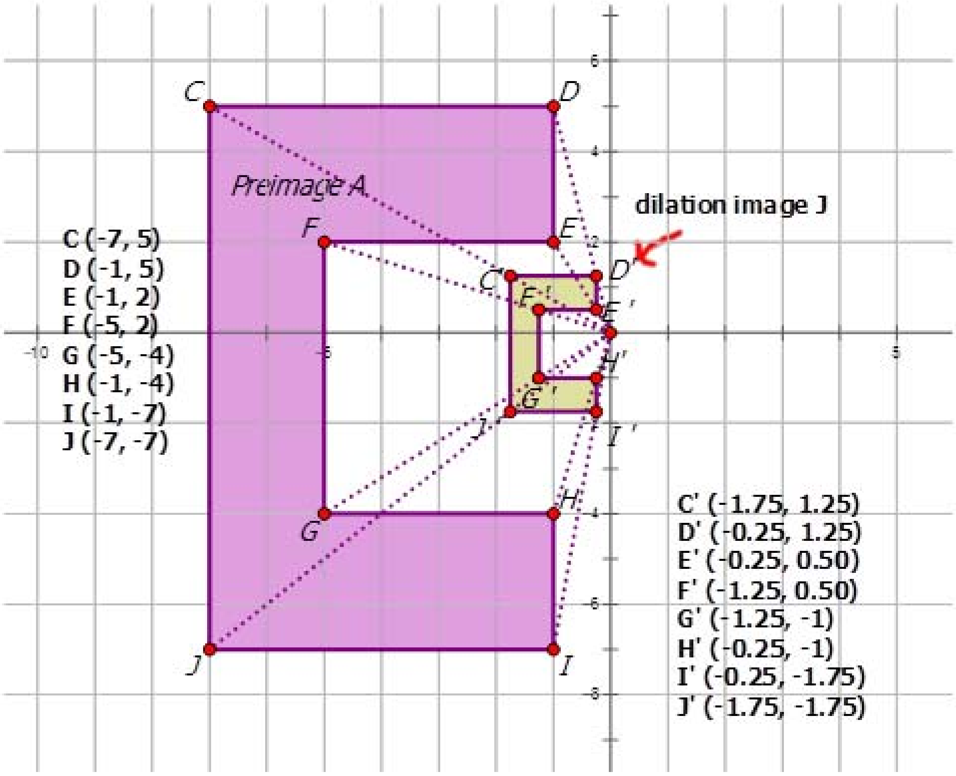
Image D	$D(-3,7)$	$E(-1,3)$	$F(-7,5)$	$G(-5,1)$
Dilation Image	$D'(-1,2.3)$	$E'(-0.3,1)$	$F'(-2.3,1.7)$	$G'(-1.7,0.3)$

The dilation image looks like the following:



Example C

Write the notation that represents the dilation of the preimage A to the dilation image J in the diagram below.



Solution: First, pick a point in the diagram to use to see how it has been affected by the dilation.

$C : (-7, 5) \quad C' : (-1.75, 1.25)$

Notice how both the x- and y-coordinates are multiplied by $\frac{1}{4}$. This indicates that the preimage A undergoes a dilation about the origin by a scale factor of $\frac{1}{4}$ to form the dilation image J. Therefore the mapping notation is $(x, y) \rightarrow \left(\frac{1}{4}x, \frac{1}{4}y\right)$.

Vocabulary

Notation Rule

A ***notation rule*** has the following form $D_k(x,y) = (kx,ky)$ and tells you that the preimage has undergone a dilation about the origin by scale factor k. If k is greater than one, the dilation image will be larger than the preimage. If k is between 0 and 1, the dilation image will be smaller than the preimage. If k is equal to 1, you will have a dilation image that is congruent to the preimage. The mapping rule corresponding to a dilation notation would be: $(x,y) \rightarrow (kx,ky)$

Center Point

The ***center point*** is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Scale Factor

The ***scale factor*** determines how much bigger or smaller the dilation image will be compared to the preimage. The scale factor often uses the symbol r.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

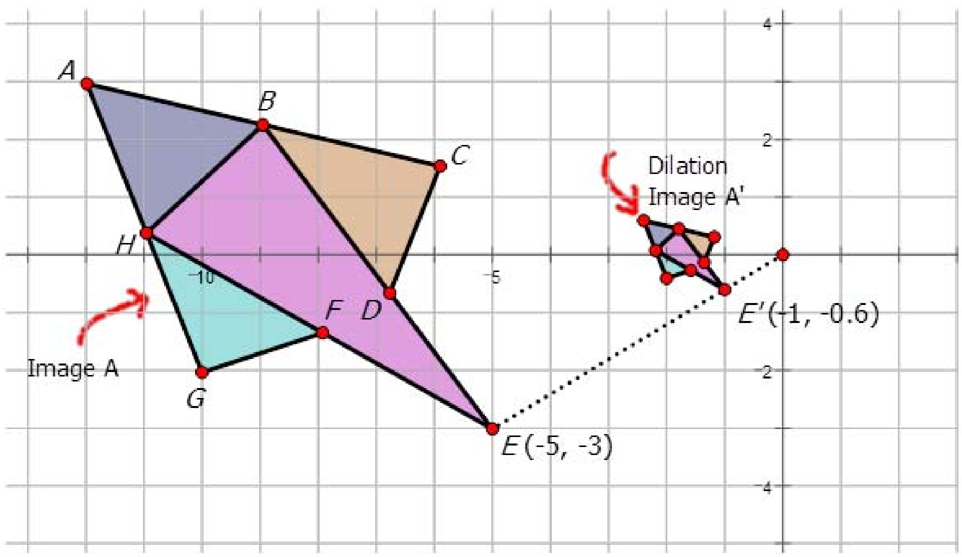
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Guided Practice

1. Thomas describes a dilation of point JT with vertices $J(-2,6)$ to $T(6,2)$ to point $J'T'$ with vertices $J'(-4,12)$ and $T'(12,4)$. Write the notation to describe this dilation for Thomas.

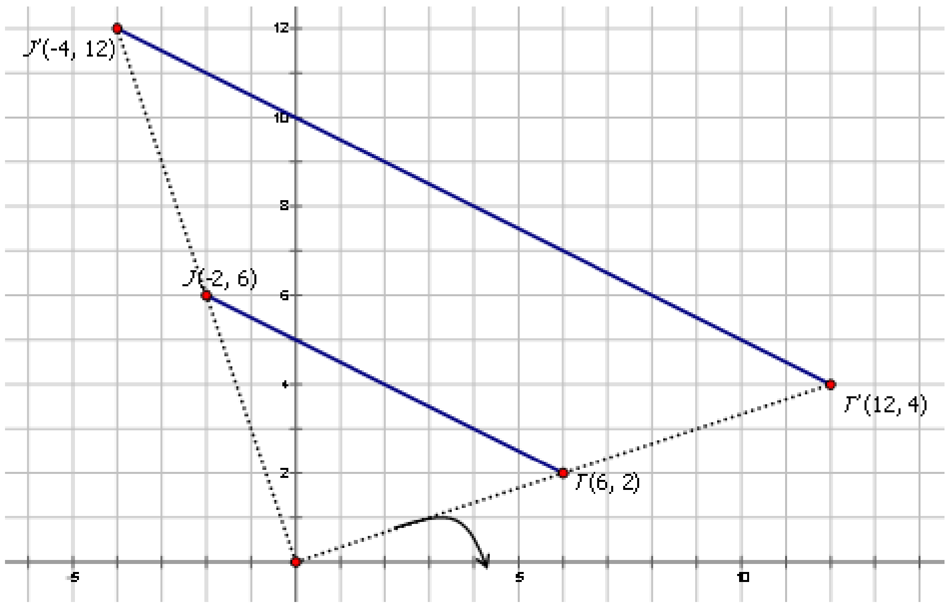
2. Given the points $A(12,8)$ and $B(8,4)$ on a line undergoing a dilation to produce $A'(6,4)$ and $B'(4,2)$, write the notation that represents the dilation.

3. Janet was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



Answers:

1.



Since the x- and y-coordinates are each multiplied by 2, the *scale factor* is 2. The mapping notation is: $(x,y) \rightarrow (2x,2y)$

2. In order to write the notation to describe the dilation, choose one point on the preimage and then the corresponding point on the dilation image to see how the point has moved. Notice that point EA is:

$$A(12,8) \rightarrow A'(6,4)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{2}$, the dilation is about the origin has a scale factor of $\frac{1}{2}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

3. In order to write the notation to describe the dilation, choose one point on the preimage A and then the corresponding point on the dilation image A' to see how the point has changed. Notice that point E is shown in the diagram:

$$E(-5,-3) \rightarrow E'(-1,-0.6)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{5}$, the dilation is about the origin has a scale factor of $\frac{1}{5}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{5}x, \frac{1}{5}y\right)$.

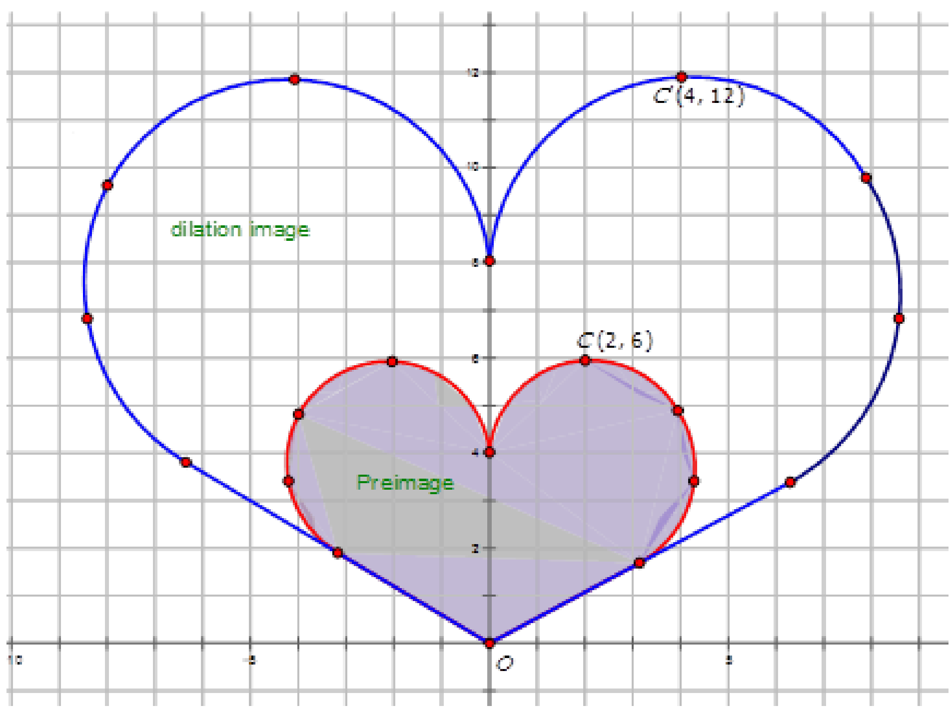
Practice

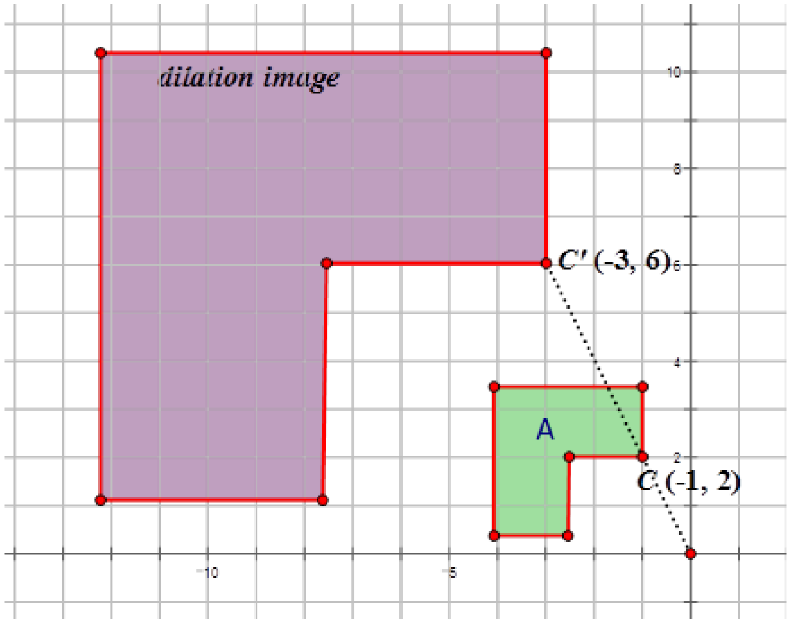
Complete the following table. Assume that the center of dilation is the origin.

TABLE 2.2:

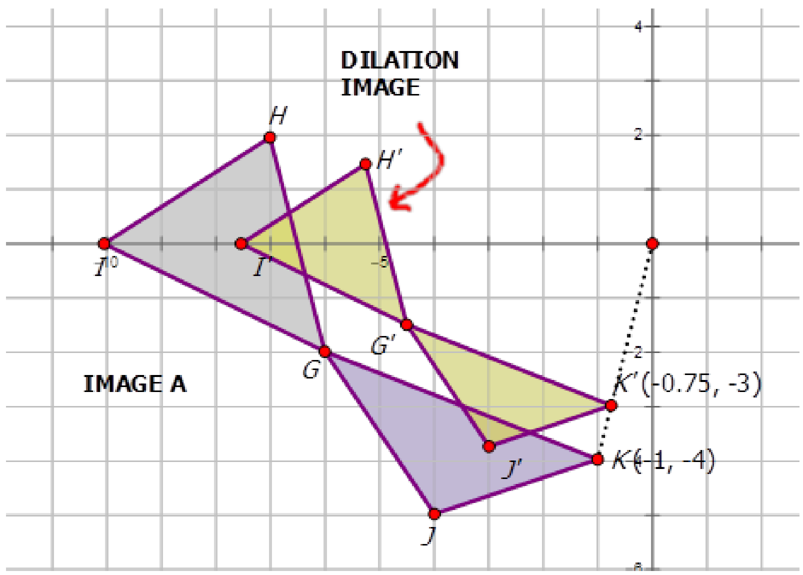
Starting Point	D_2	D_5	$D_{\frac{1}{2}}$	$D_{\frac{3}{4}}$
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				
6. (9, 4)				
7. (-1, 3)				
8. (-5, 2)				
9. (2, 6)				
10. (-5, 7)				

Write the notation that represents the dilation of the preimage to the image for each diagram below.

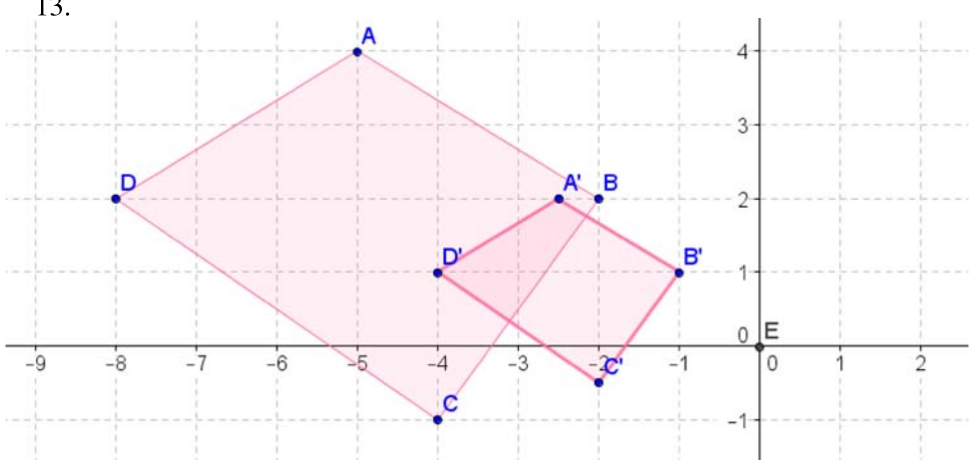
11. 


dilation image
C′ (-3, 6)
A
C (-1, 2)

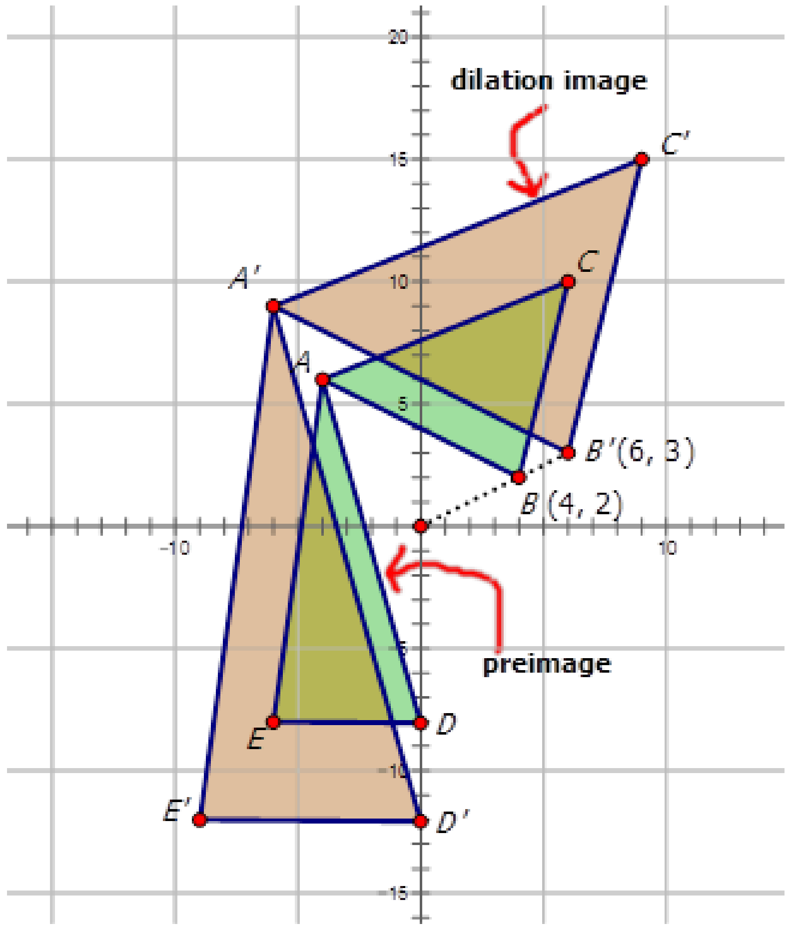
12.


DILATION
IMAGE
H
H′
I
I′
G′
G
IMAGE A
K′(-0.75, -3)
K(-1, -4)
J′
J

13.


A
D
A′
B
D′
B′
E
C′
C

14.

15. 
dilation image
C′
C
A′
A
B′(6, 3)
B (4, 2)
preimage
D
E
E′
D′
20
15
10
5
-10
10
-5
-10
-15

Concept 3 SLT 5 Develop a definition for similarity using the principles of dilation.

Have you ever built a ramp? Take a look at this dilemma.

Marc, Isaac and Isabelle thought that designing a skateboard ramp would be easy. Because of this, they have decided to build two of them in their skatepark. Using the computer, they found the measurements for the first skateboard ramp design.

It has the form of a triangle and is in three dimensions, so it also has a width. Here are the dimensions for the first ramp.

28" long $\times$ 38.5" wide $\times$ 12" high

Isaac writes the following proportion on a piece of paper.

$$\frac{28''}{14''} = \frac{38.5''}{\square} = \frac{12''}{6''}$$

"The two ramps are going to be similar, but not congruent," Isaac begins to explain.

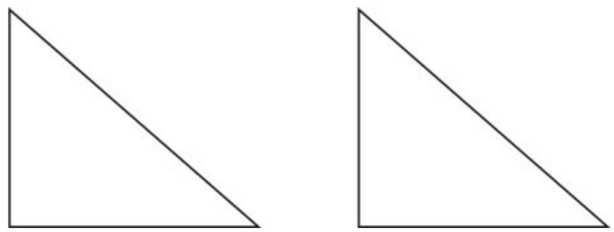
At that moment, his mom begins calling him and he dashes out the door leaving Isabelle and Marc with his work and with the proportion.

"What is the difference between similar and congruent?" Isabelle asks.

There are two problems here. One has to do with similar and congruent triangles. The other has to do with the missing measurement.

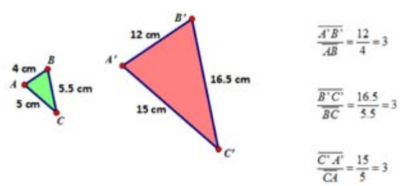
Guidance

You have heard the word ***congruent*** used regarding line segments being the same length. The word congruent can apply to other things in geometry besides lines and line segments. **Congruent means being exactly the same. When two line segments have the same length, we can say that they are congruent. When two figures have the same shape and size, we can say that the two figures are congruent.**



These two triangles are congruent. They are exactly the same in every way. They are the same size and the same shape. We can also say that their side lengths are the same and that their angle measures are the same.

When a dilation is performed on a triangle, or any other figure, the preimage (original figure) and image (dilated figure) are no longer congruent. The corresponding sides of the triangles are proportional. However, corresponding angles are equal.



There are five properties of figures that remain the same between the preimage and image under a dilation.

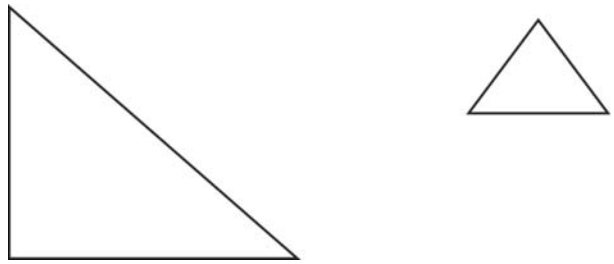
5 Properties Preserved under a Dilation

1. Angle measurements
2. Parallelism
3. Colinearity
4. Midpoint
5. Orientation

In a dilation, the two figures will be *similar* . Similar means that the figures have the same shape, but not the same size. In similar figures, corresponding sides are proportional and corresponding angles are equal. The symbol ~ means similar. Similar figures are not congruent. Therefore, $\triangle ABC \sim \triangle A'B'C'$ **.**

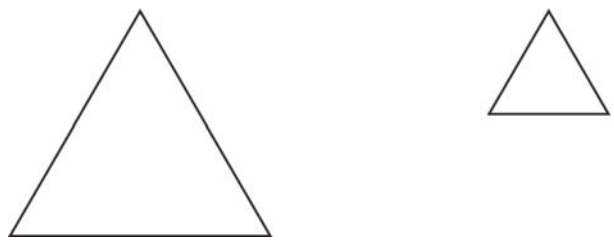
Identify the following triangles as congruent, similar or neither.

Example A



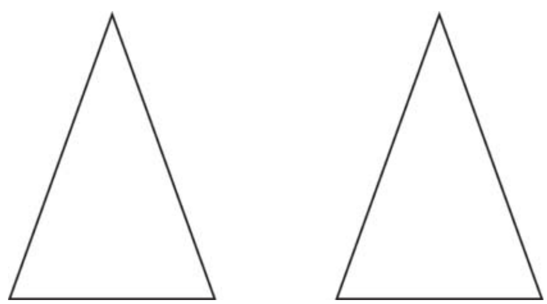
Solution: Neither

Example B



Solution: Similar

Example C



Solution: Congruent

Now let's go back to the question about the ramps. Here is the original problem once again.

Marc, Isaac and Isabelle thought that designing a skateboard ramp would be easy. Because of this, they have decided to build two of them in their skatepark. Using the computer, they found the measurements for the first skateboard ramp design.

It has the form of a triangle and is in three dimensions, so it also has a width. Here are the dimensions for the first ramp.

28" long $\times$ 38.5" wide $\times$ 12" high

Isaac writes the following proportion on a piece of paper.

$$\frac{28''}{14''} = \frac{38.5''}{\square} = \frac{12''}{6''}$$

"The two ramps are going to be similar, but not congruent," Isaac begins to explain.

At that moment, his mom begins calling him and he dashes out the door leaving Isabelle and Marc with his work and with the proportion.

"What is the difference between similar and congruent?" Isabelle asks.

Let's review the difference between similar figures and congruent figures. Solving the proportion will be in another Concept.

A similar figure is one that is the same shape but a different size from the original one. The measurements of similar figures have a relationship. They are proportional. In other words, their dimensions form a proportion.

Congruent figures are the same size and shape exactly. Congruent figures would have the same measurements.

The ramp dimensions are similar. Isaac left Marc and Isabelle with that much information, which means that the dimensions of the ramps are proportional but not exact.

Vocabulary

Congruent

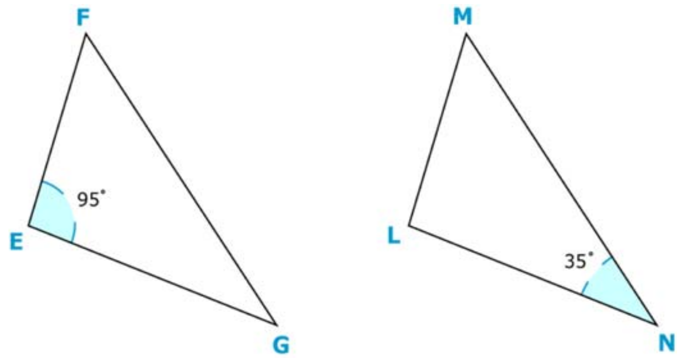
having the same size and shape and measurement

Similar

having the same shape, but not the same size. Similar shapes are proportional to each other.

Guided Practice

Here is one for you to try on your own.



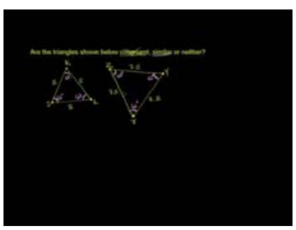
Are there two triangles similar, congruent or neither?

Answer

When you look at these two triangles, you can see that they are exactly alike. The problem could be misleading if you look at the angle measures, but notice that different angle measures have been given.

These two triangles are congruent.

Video Review



MEDIA

Click image to the left for use the URL below.

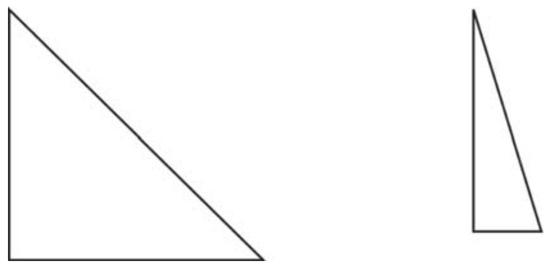
URL: http://www.ck12.org/flx/render/embeddedobject/5434

Khan Academy Congruent and Similar Triangles

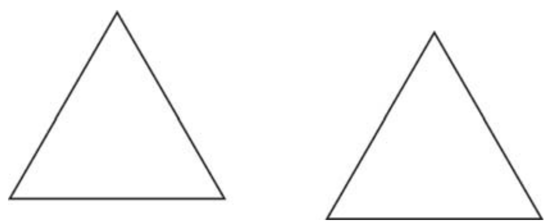
Practice

<u>Directions:</u> Identify the given triangles as visually similar, congruent or neither.

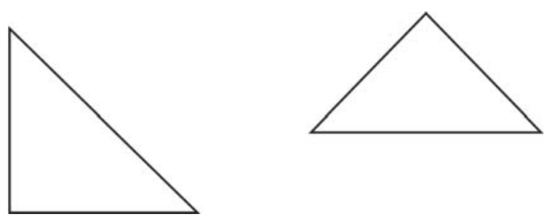
1.



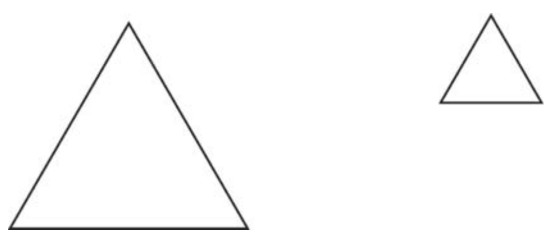
2.



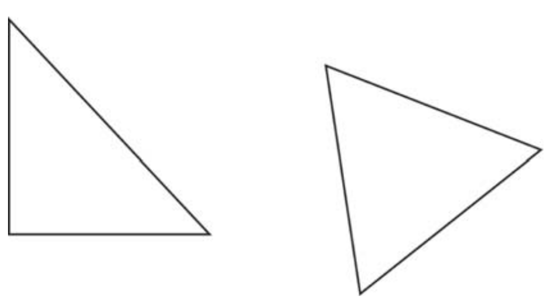
3.



4.



5.



Directions: Answer each of the following questions.

6. Triangles ABC and DEF are congruent. Does this mean that their angle measures are the same? Why?

7. True or false. If triangles DEF and GHI are similar, then the side lengths are different but the angle measures are the same.

8. True or false. Similar figures have exactly the same size and shape.

9. True or false. Congruent figures are exactly the same in every way.

10. Triangles LMN and HIJ are similar. If this is true, then the side lengths are the same, true or false.

11. True or false. To figure out if two figures are similar, then their side lengths form a proportion.

12. Define similar figures

13. Define congruent figures.

14. Use a ruler to draw a congruent pair of triangles.

15. Use a ruler to draw a pair of triangles that is similar.

Determine if the following statements are true or false.

16. All equilateral triangles are similar.

17. All isosceles triangles are similar.

18. All rectangles are similar.

19. All rhombuses are similar.

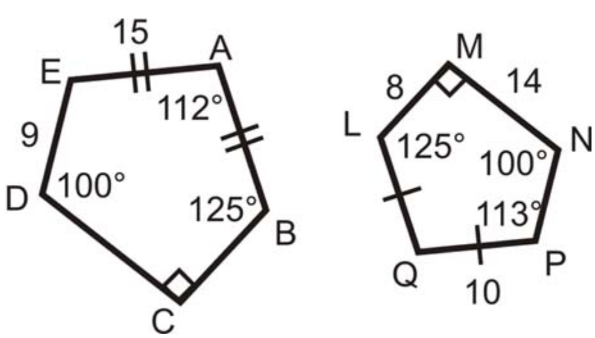
20. All squares are similar.

21. All congruent polygons are similar.

22. All similar polygons are congruent.

23. All regular pentagons are similar.

Use the picture to the right to answer questions 24-28.



24. Find $m\angle E$ and $m\angle Q$.

25. $ABCDE \sim QLMNP$, Find the scale factor.

26. Find BC.

27. Find CD.

28. Find NP.

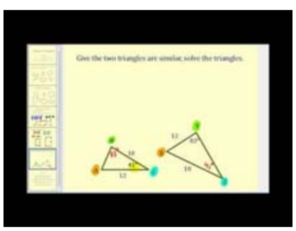
CONCEPT 4 SLT 6 Develop similarity statements and identify corresponding angles and sides based on statements.

What if you were comparing a baseball diamond and a softball diamond? A baseball diamond is a square with 90 foot sides. A softball diamond is a square with 60 foot sides. Are the two diamonds similar? If so, what is the scale factor? After completing this Concept, you'll be able to use your knowledge of similar polygons to answer these questions.

Watch This

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/52542

CK-12 Foundation: Chapter7SimilarPolygonsandScaleFactorsA



MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1346

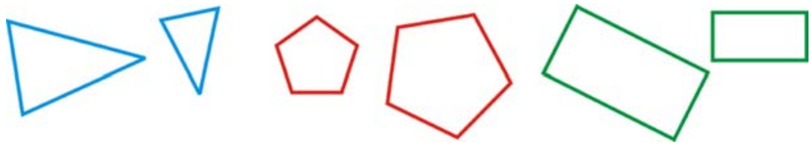
James Sousa: Similar Polygons

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1347

James Sousa: Scale Factor

Guidance

Similar polygons are two polygons with the same shape, but not necessarily the same size. Similar polygons have corresponding angles that are **congruent,** and corresponding sides that are **proportional.**



These polygons are not similar:

Think about similar polygons as enlarging or shrinking the same shape. The symbol $\sim$ is used to represent similarity. Specific types of triangles, quadrilaterals, and polygons will always be similar. For example, ***all equilateral triangles are similar*** and ***all squares are similar.*** If two polygons are similar, we know the lengths of corresponding sides are proportional. In similar polygons, the ratio of one side of a polygon to the corresponding side of the other is called the **scale factor**. The ratio of all parts of a polygon (including the perimeters, diagonals, medians, midsegments, altitudes) is the same as the ratio of the sides.

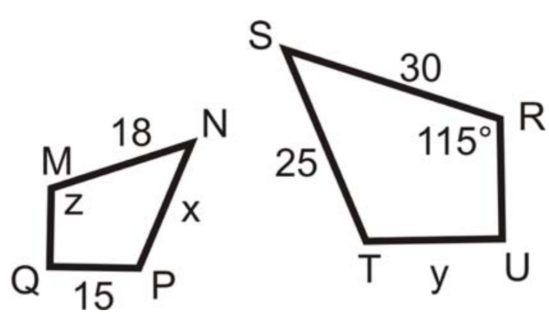
Example A

Suppose $\triangle ABC \sim \triangle JKL$. Based on the similarity statement, which angles are congruent and which sides are proportional?

Just like in a congruence statement, the congruent angles line up within the similarity statement. So, $\angle A \cong \angle J, \angle B \cong \angle K$, and $\angle C \cong \angle L$. Write the sides in a proportion: $\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$. Note that the proportion could be written in different ways. For example, $\frac{AB}{BC} = \frac{JK}{KL}$ is also true.

Example B

$MNPQ \sim RSTU$. What are the values of x, y and z?

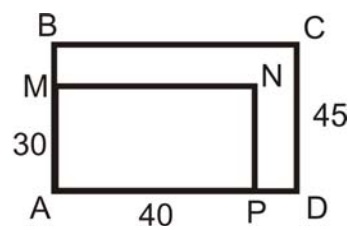


In the similarity statement, $\angle M \cong \angle R$, so $z = 115^\circ$. For x and y, set up proportions.

$$\begin{aligned} \frac{18}{30} &= \frac{x}{25} \\ 450 &= 30x \\ x &= 15 \end{aligned} \qquad \begin{aligned} \frac{18}{30} &= \frac{15}{y} \\ 18y &= 450 \\ y &= 25 \end{aligned}$$

Example C

$ABCD \sim AMNP$. Find the scale factor and the length of BC.



Line up the corresponding sides, AB and $AM = CD$, so the scale factor is $\frac{30}{45} = \frac{2}{3}$ or $\frac{3}{2}$. Because BC is in the bigger rectangle, we will multiply 40 by $\frac{3}{2}$ because $\frac{3}{2}$ is greater than 1. $BC = \frac{3}{2}(40) = 60$.

Watch this video for help with the Examples above.

MEDIA

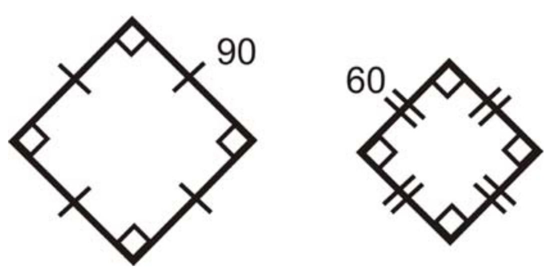
Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52543

CK-12 Foundation: Chapter7SimilarPolygonsandScaleFactorsB

Concept Problem Revisited

All of the sides in the baseball diamond are 90 feet long and 60 feet long in the softball diamond. This means all the sides are in a $\frac{90}{60} = \frac{3}{2}$ ratio. All the angles in a square are congruent, all the angles in both diamonds are congruent. The two squares are similar and the scale factor is $\frac{3}{2}$.

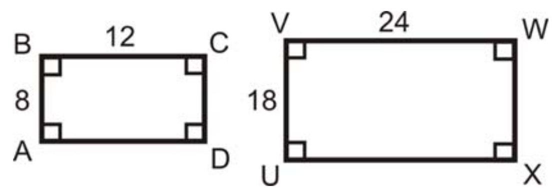


Vocabulary

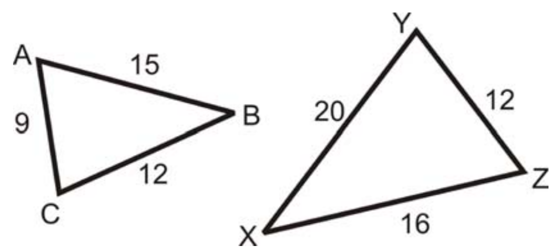
Similar polygons are two polygons with the same shape, but not necessarily the same size. The corresponding angles of similar polygons are ***congruent*** (exactly the same) and the corresponding sides are ***proportional*** (in the same ratio). In similar polygons, the ratio of one side of a polygon to the corresponding side of the other is called the ***scale factor***.

Guided Practice

1. $ABCD$ and $UVWX$ are below. Are these two rectangles similar?



2. What is the scale factor of $\triangle ABC$ to $\triangle XYZ$? Write the similarity statement.



3. $\triangle ABC \sim \triangle MNP$. The perimeter of $\triangle ABC$ is 150, $AB = 32$ and $MN = 48$. Find the perimeter of $\triangle MNP$.

Answers:

1. All the corresponding angles are congruent because the shapes are rectangles.

Let's see if the sides are proportional. $\frac{8}{12} = \frac{2}{3}$ and $\frac{18}{24} = \frac{3}{4}$. $\frac{2}{3} \neq \frac{3}{4}$, so the sides are ***not*** in the same proportion, and the rectangles are ***not*** similar.

2. All the sides are in the same ratio. Pick the two largest (or smallest) sides to find the ratio.

$$\frac{15}{20} = \frac{3}{4}$$

For the similarity statement, line up the proportional sides. $AB \rightarrow XY, BC \rightarrow XZ, AC \rightarrow YZ$, so $\triangle ABC \sim \triangle YXZ$.

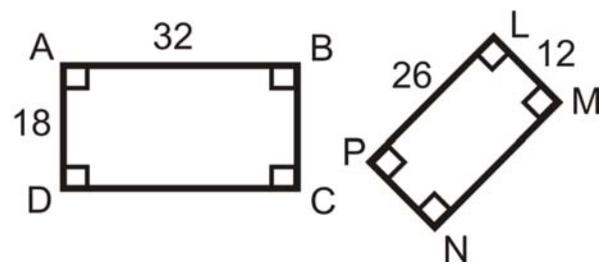
3. From the similarity statement, AB and MN are corresponding sides. The scale factor is $\frac{32}{48} = \frac{2}{3}$ or $\frac{3}{2}$. $\triangle ABC$ is the smaller triangle, so the perimeter of $\triangle MNP$ is $\frac{3}{2}(150) = 225$.

Explore More

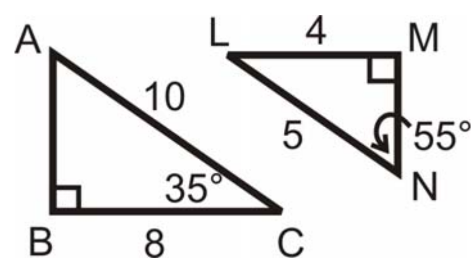
1. $\triangle BIG \sim \triangle HAT$. List the congruent angles and proportions for the sides.
2. If $BI = 9$ and $HA = 15$, find the scale factor.
3. If $BG = 21$, find HT.
4. If $AT = 45$, find IG.
5. Find the perimeter of $\triangle BIG$ and $\triangle HAT$. What is the ratio of the perimeters?

Determine if the following triangles and quadrilaterals are similar. If they are, write the similarity statement.

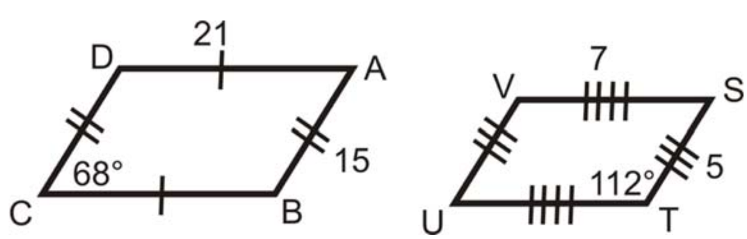
6.



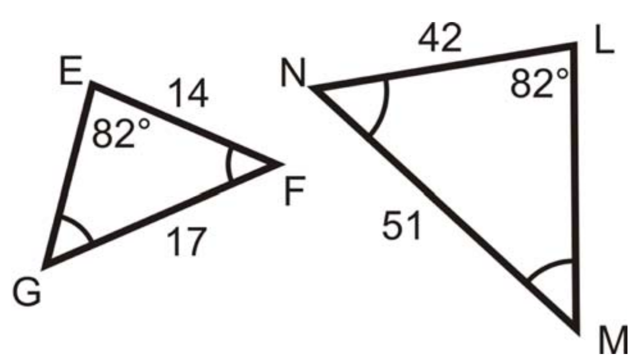
7.



8.

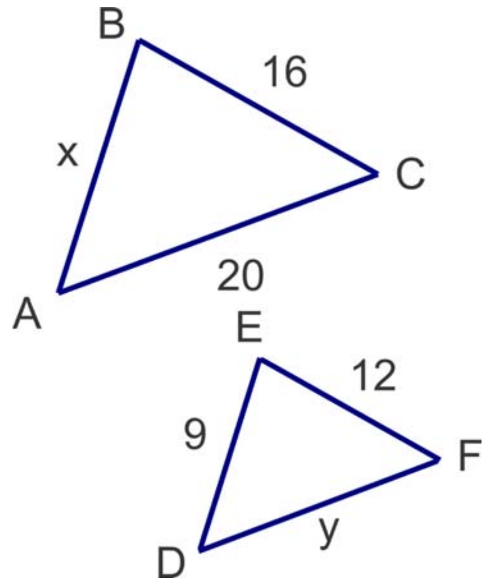


9.

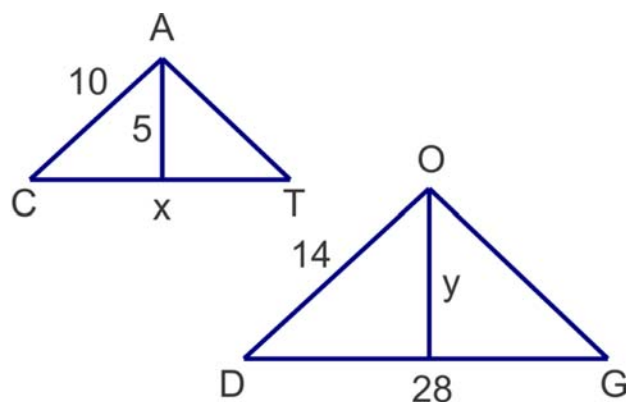


Solve for x and y.

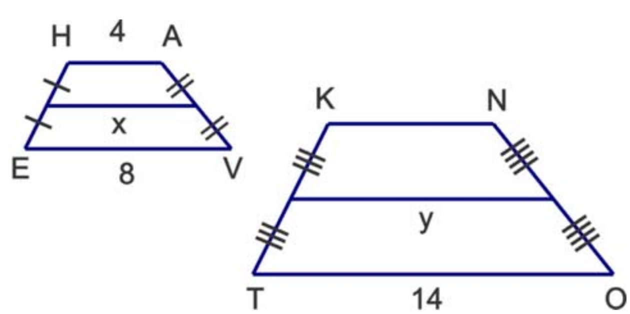
10. $\triangle ABC \sim \triangle DEF$



11. $\triangle CAT \sim \triangle DOG$

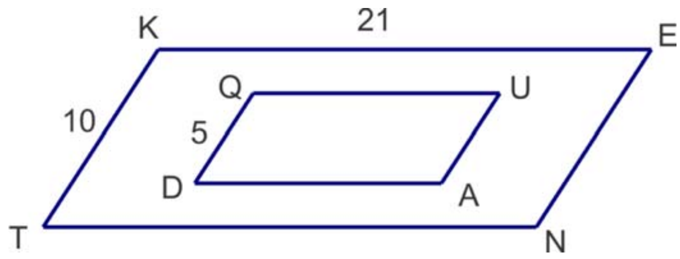


12. Trapezoids $HAVE \sim KNOT$

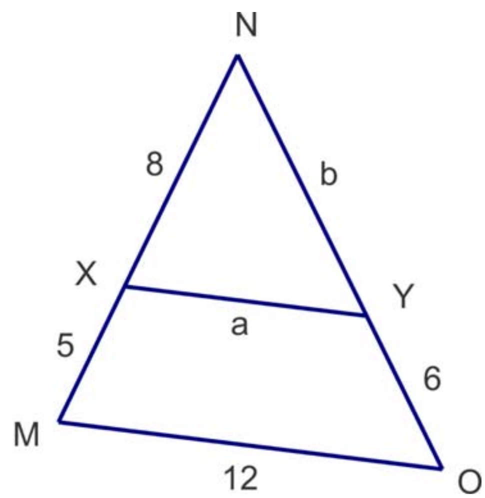


Answer the following.

13. Find the perimeter of $QUAD$ given $QUAD \sim KENT$.



14. Solve for a and b given $\triangle MNO \sim \triangle XNY$.



15. Two similar octagons have a scale factor of $\frac{9}{11}$. If the perimeter of the smaller octagon is 99 meters, what is the perimeter of the larger octagon?

16. Two right triangles are similar. The legs of one of the triangles are 5 and 12. The second right triangle has a hypotenuse of length 39. What is the scale factor between the two triangles?

CONCEPT 5 SLT 7 Show that AA, SAS, and SSS are sufficient conditions to prove triangle similarity.

What if you were given a pair of triangles and asked if they were similar? How would you know?

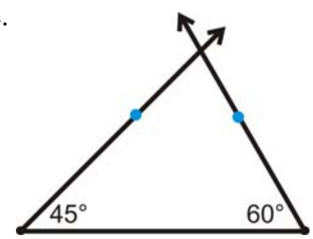
Guidance

The Third Angle Theorem states if two angles are congruent to two angles in another triangle, the third angles are congruent too. Because a triangle has 180°, the third angle in any triangle is 180° minus the other two angle measures. Let's investigate what happens when two different triangles have the same angle measures.

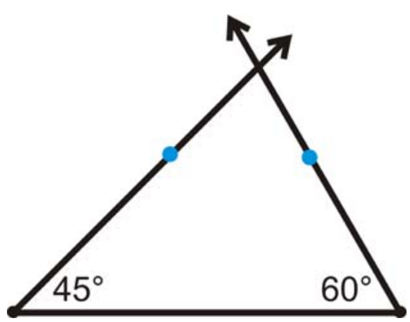
Investigation: Constructing Similar Triangles

Tools Needed: pencil, paper, protractor, ruler

1. Draw a 45° angle. Extend the horizontal side and then draw a 60° angle on the other side of this side. Extend the other side of the 45° angle and the 60° angle so that they intersect to form a triangle. What is the measure of the third angle? Measure the length of each side.



2. Repeat Step 1 and make the horizontal side between the 45° and 60° angle at least 1 inch longer than in Step 1. This will make the entire triangle larger. Find the measure of the third angle and measure the length of each side.

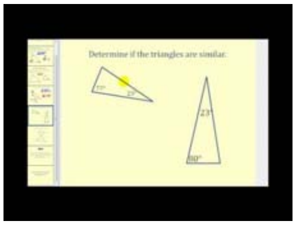


3. Find the ratio of the sides. Put the sides opposite the 45° angles over each other, the sides opposite the 60° angles over each other, and the sides opposite the third angles over each other. What happens?

AA Similarity Postulate: If two angles in one triangle are congruent to two angles in another triangle, the two triangles are similar.

The AA Similarity Postulate is a shortcut for showing that two ***triangles*** are similar. If you know that two angles in one triangle are congruent to two angles in another, which is now enough information to show that the two triangles are similar. Then, you can use the similarity to find the lengths of the sides

Watch This



MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/1357

James Sousa: Similar Triangles by AA

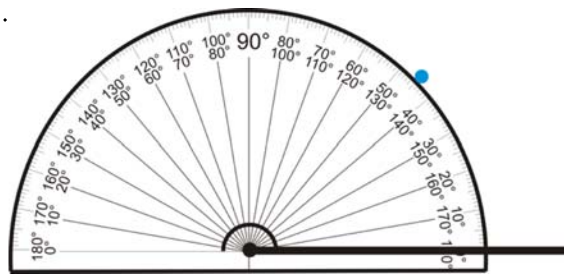
Guidance

If we know that two sides are proportional AND the included angles are congruent, then are the two triangles are similar? Let's investigate.

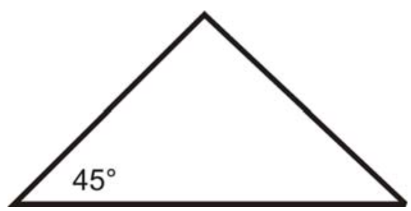
Investigation: SAS Similarity

Tools Needed: paper, pencil, ruler, protractor, compass

1. Construct a triangle with sides 6 cm and 4 cm and the *included* angle is 45°.



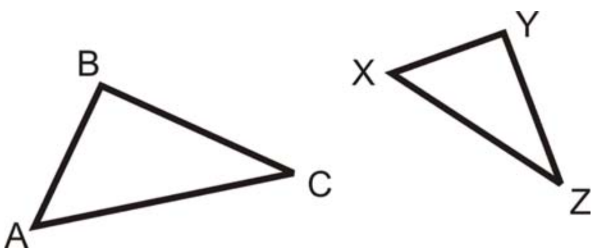
2. Repeat Step 1 and construct another triangle with sides 12 cm and 8 cm and the included angle is 45°.
3. Measure the other two angles in both triangles. What do you notice?



4. Measure the third side in each triangle. Make a ratio. Is this ratio the same as the ratios of the sides you were given?

SAS Similarity Theorem: If two sides in one triangle are proportional to two sides in another triangle and the included angle in the first triangle is congruent to the included angle in the second, then the two triangles are similar.

In other words, if $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A \cong \angle X$, then $\triangle ABC \sim \triangle XYZ$.

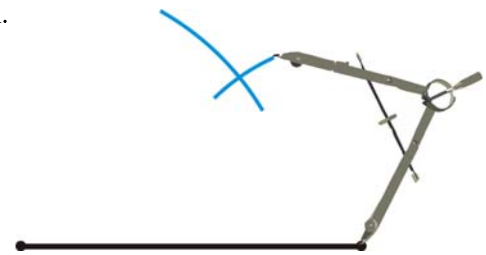


If you do not know any angle measures, can you say two triangles are similar? Let's investigate this to see.

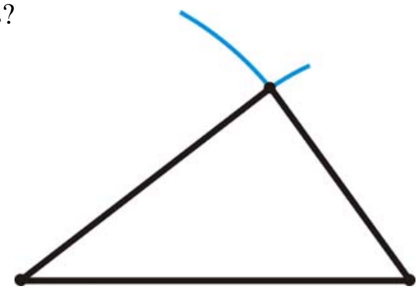
Investigation: SSS Similarity

Tools Needed: ruler, compass, protractor, paper, pencil

1. Construct a triangle with sides 6 cm, 8 cm, and 10 cm.



2. Construct a second triangle with sides 9 cm, 12 cm, and 15 cm.
3. Using your protractor, measure the angles in both triangles. What do you notice?
4. Line up the corresponding sides. Write down the ratios of these sides. What happens?

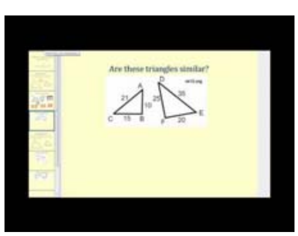


To see an animated construction of this, click: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.html

From #3, you should notice that the angles in the two triangles are equal. Second, when the corresponding sides are lined up, the sides are all in the same proportion, $\frac{6}{9} = \frac{8}{12} = \frac{10}{15}$. If you were to repeat this activity, for a 3-4-5 or 12-16-20 triangle, you will notice that they are all similar. That is because, each of these triangles are multiples of 3-4-5. If we generalize what we found in this investigation, we have the SSS Similarity Theorem.

SSS Similarity Theorem: If the corresponding sides of two triangles are proportional, then the two triangles are similar.

Watch This



MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/1349

James Sousa: Similar Triangles Using SSS andSAS

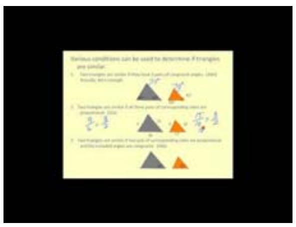
Vocabulary

Two triangles are ***similar*** if all their corresponding angles are ***congruent*** (exactly the same) and their corresponding sides are ***proportional*** (in the same ratio).

CONCEPT 6 SLT 8 Use triangle similarity criteria (AA, SAS, SSS) to show that two triangles are similar.

What if you were given a pair of triangles? How could you determine if the two triangles are similar?

Watch This



MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/1302

James Sousa: Similar Triangles

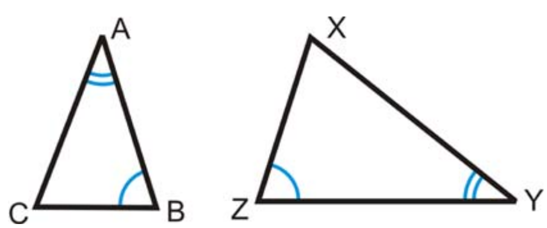
Guidance

By definition, two triangles are similar if all their corresponding angles are congruent and their corresponding sides are proportional. It is not necessary to check all angles and sides in order to tell if two triangles are similar. In fact, if you only know that

1. Two pairs of corresponding angles are congruent, **AA Similarity Postulate**,
2. Two pairs of sides are proportional and their included angles are congruent, **SAS Similarity Theorem**, or
3. All sides are proportional, **SSS Similarity Theorem**

to know that the triangles are similar.

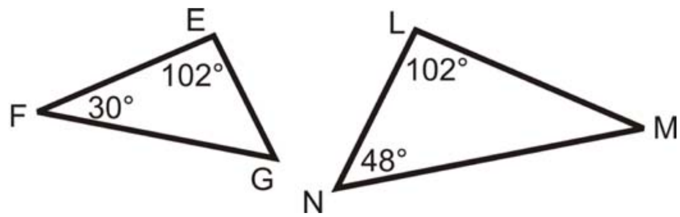
AA Similarity Postulate: If two angles in one triangle are congruent to two angles in another triangle, then the two triangles are similar.



If $\angle A \cong \angle Y$ and $\angle B \cong \angle Z$, then $\triangle ABC \sim \triangle YZX$.

Example A

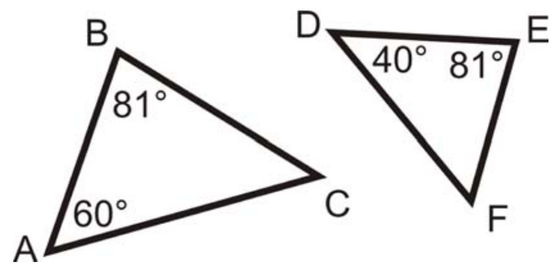
Determine if the following two triangles are similar. If so, write the similarity statement.



Compare the angles to see if we can use the AA Similarity Postulate. Using the Triangle Sum Theorem, $m\angle G = 48^\circ$ and $m\angle M = 30^\circ$ So, $\angle F \cong \angle M, \angle E \cong \angle L$ and $\angle G \cong \angle N$ and the triangles are similar. $\triangle FEG \sim \triangle MLN$.

Example B

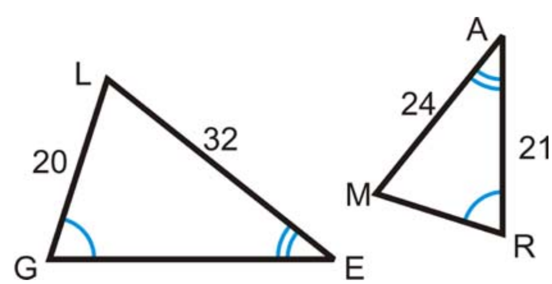
Determine if the following two triangles are similar. If so, write the similarity statement.



Compare the angles to see if we can use the AA Similarity Postulate. Using the Triangle Sum Theorem, $m\angle C = 39^\circ$ and $m\angle F = 59^\circ$. $m\angle C \neq m\angle F$, So $\triangle ABC$ and $\triangle DEF$ are not similar.

Example C

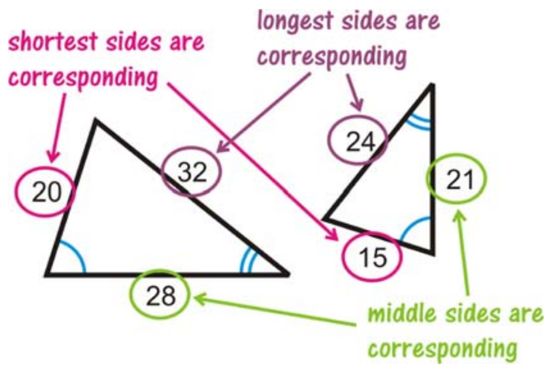
$\triangle LEG \sim \triangle MAR$ by AA. Find GE and MR.



Set up a proportion to find the missing sides.

$$\begin{aligned} \frac{24}{32} &= \frac{MR}{20} \\ 480 &= 32MR \\ 15 &= MR \end{aligned} \qquad \begin{aligned} \frac{24}{32} &= \frac{21}{GE} \\ 24GE &= 672 \\ GE &= 28 \end{aligned}$$

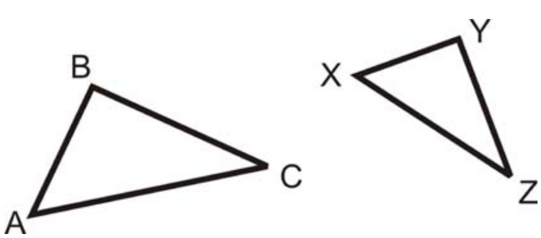
When two triangles are similar, the corresponding sides are proportional. But, what are the corresponding sides? Using the triangles from this example, we see how the sides line up in the diagram to the right.



MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/80497

CK-12 Foundation: AA Similarity

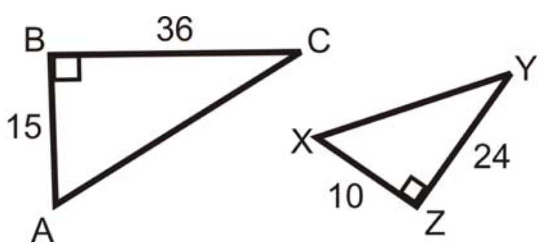
SAS Similarity Theorem: If two sides in one triangle are proportional to two sides in another triangle and the included angle in both are congruent, then the two triangles are similar.



If $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A \cong \angle X$, then $\triangle ABC \sim \triangle XYZ$.

Example D

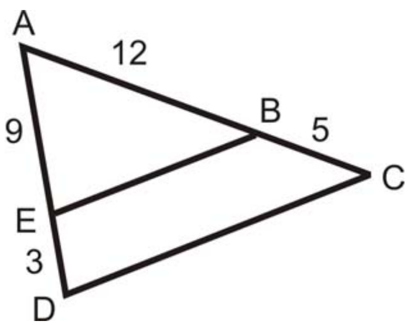
Are the two triangles similar? How do you know?



We know that $\angle B \cong \angle Z$ because they are both right angles and $\frac{10}{15} = \frac{24}{36}$. So, $\frac{AB}{XZ} = \frac{BC}{ZY}$ and $\triangle ABC \sim \triangle XZY$ by SAS.

Example E

Are there any similar triangles in the figure? How do you know?



$\angle A$ is shared by $\triangle EAB$ and $\triangle DAC$, so it is congruent to itself. Let's see if $\frac{AE}{AD} = \frac{AB}{AC}$.

$$\frac{9}{9+3} = \frac{12}{12+5}$$
$$\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$$

The two triangles are *not* similar.

Example F

From Example B, what should BC equal for $\triangle EAB \sim \triangle DAC$?

The proportion we ended up with was $\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$. AC needs to equal 16, so that $\frac{12}{16} = \frac{3}{4}$. $AC = AB + BC$ and $16 = 12 + BC$. BC should equal 4.

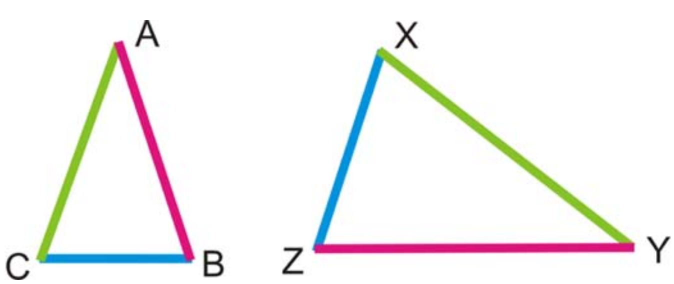
MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/80499

CK-12 Foundation: SAS Similarity

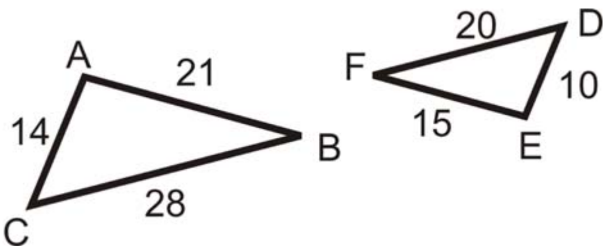
SSS Similarity Theorem: If all three pairs of corresponding sides of two triangles are proportional, then the two triangles are similar.



If $\frac{AB}{YZ} = \frac{BC}{ZX} = \frac{AC}{XY}$, then $\triangle ABC \sim \triangle YZX$.

Example G

Determine if the following triangles are similar. If so, explain why and write the similarity statement.



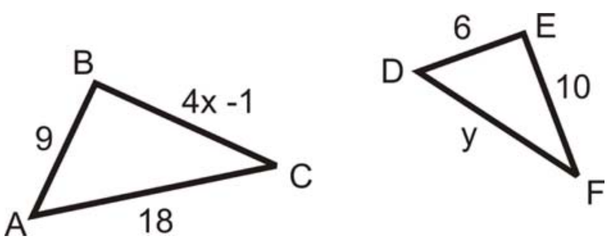
We will need to find the ratios for the corresponding sides of the triangles and see if they are all the same. Start with the longest sides and work down to the shortest sides.

$\frac{BC}{FD} = \frac{28}{20} = \frac{7}{5}$
$\frac{BA}{FE} = \frac{21}{15} = \frac{7}{5}$
$\frac{AC}{ED} = \frac{14}{10} = \frac{7}{5}$

Since all the ratios are the same, $\triangle ABC \sim \triangle EFD$ by the SSS Similarity Theorem.

Example H

Find x and y, such that $\triangle ABC \sim \triangle DEF$.



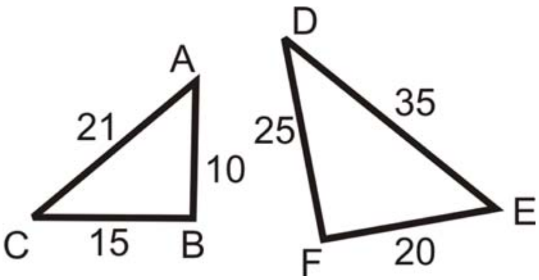
According to the similarity statement, the corresponding sides are: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. Substituting in what we know, we have $\frac{9}{6} = \frac{4x-1}{10} = \frac{18}{y}$.

$$\begin{aligned}
\frac{9}{6} &= \frac{4x-1}{10} \\
9(10) &= 6(4x-1) \\
90 &= 24x-6 \\
96 &= 24x \\
x &= 4
\end{aligned}$$

$$\begin{aligned}
\frac{9}{6} &= \frac{18}{y} \\
9y &= 18(6) \\
9y &= 108 \\
y &= 12
\end{aligned}$$

Example I

Determine if the following triangles are similar. If so, explain why and write the similarity statement.



We will need to find the ratios for the corresponding sides of the triangles and see if they are all the same. Start with the longest sides and work down to the shortest sides.

$\frac{AC}{ED} = \frac{21}{35} = \frac{3}{5}$

$\frac{BC}{FD} = \frac{15}{25} = \frac{3}{5}$

$\frac{AB}{EF} = \frac{10}{20} = \frac{1}{2}$

Since the ratios are not all the same, the triangles are not similar.

MEDIA

Click image to the left for use the URL below.

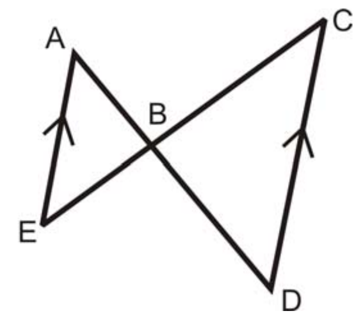
URL: http://www.ck12.org/flx/render/embeddedobject/80505

CK-12 Foundation: SSS Similarity

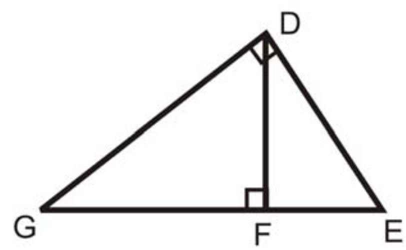
Guided Practice

For #1 - 6, determine if the following triangles are similar. If so, explain why they are similar (AA, SAS, or SSS similarity) and write the similarity statement.

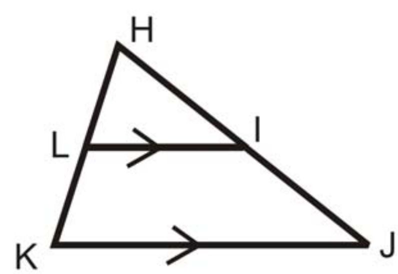
1.



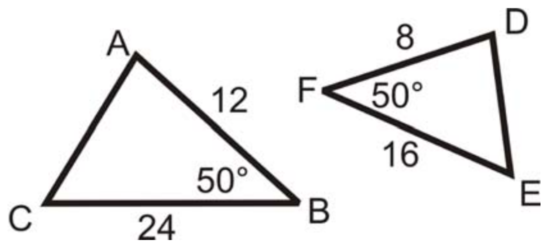
2.



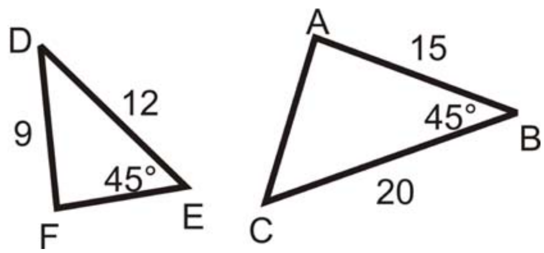
3.



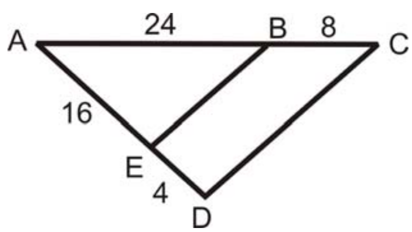
4.



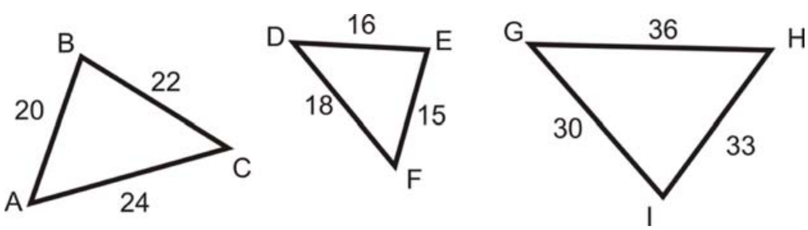
5.



6.



Use the triangles below for #7 - 9.



7. Is $\triangle ABC \sim \triangle DEF$?

8. Is $\triangle DEF \sim \triangle GHI$?

9. Is $\triangle ABC \sim \triangle GHI$?

Answers:

1. Because $\overline{AE} \| \overline{CD}, \angle A \cong \angle D$ and $\angle C \cong \angle E$ by the Alternate Interior Angles Theorem. By the AA Similarity Postulate, $\triangle ABE \sim \triangle DBC$.

2. Yes, there are three similar triangles that each have a right angle. $DGE \sim FGD \sim FDE$.

3. By the reflexive property, $\angle H \cong \angle H$. Because the horizontal lines are parallel, $\angle L \cong \angle K$ (corresponding angles). So yes, there is a pair of similar triangles. $HLI \sim HKJ$.

4. We can see that $\angle B \cong \angle F$ and these are both included angles. We just have to check that the sides around the angles are proportional.

$\frac{AB}{DF} = \frac{12}{8} = \frac{3}{2}$
$\frac{BC}{FE} = \frac{24}{16} = \frac{3}{2}$
Since the ratios are the same $\triangle ABC \sim \triangle DFE$ by the SAS Similarity Theorem.

5. The triangles are not similar because the angle is not the included angle for both triangles.

6. $\angle A$ is the included angle for both triangles, so we have a pair of congruent angles. Now we have to check that the sides around the angles are proportional.

$\frac{AE}{AD} = \frac{16}{16+4} = \frac{16}{20} = \frac{4}{5}$
$\frac{AB}{AC} = \frac{24}{24+8} = \frac{24}{32} = \frac{3}{4}$
The ratios are not the same so the triangles are not similar.

7. $\triangle ABC$ and $\triangle DEF$: Is $\frac{20}{15} = \frac{22}{16} = \frac{24}{18}$?

Reduce each fraction to see if they are equal. $\frac{20}{15} = \frac{4}{3}, \frac{22}{16} = \frac{11}{8}$, and $\frac{24}{18} = \frac{4}{3}$.

$\frac{4}{3} \neq \frac{11}{8}, \triangle ABC$ and $\triangle DEF$ are **not** similar.

8. $\triangle DEF$ and $\triangle GHI$: Is $\frac{15}{30} = \frac{16}{33} = \frac{18}{36}$?

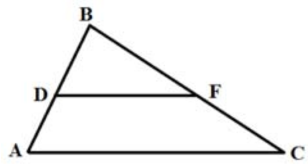
$\frac{15}{30} = \frac{1}{2}, \frac{16}{33} = \frac{16}{33}$, and $\frac{18}{36} = \frac{1}{2}$. $\frac{1}{2} \neq \frac{16}{33}, \triangle DEF$ is **not** similar to $\triangle GHI$.

9. $\triangle ABC$ and $\triangle GHI$: Is $\frac{20}{30} = \frac{22}{33} = \frac{24}{36}$?

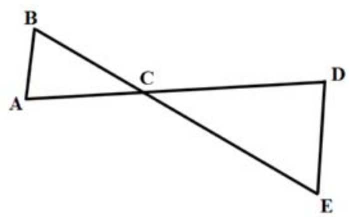
$\frac{20}{30} = \frac{2}{3}, \frac{22}{33} = \frac{2}{3}$, and $\frac{24}{36} = \frac{2}{3}$. All three ratios reduce to $\frac{2}{3}$, $\triangle ABC \sim \triangle GIH$.

CONCEPT 7 SLT 9 Use triangle similarity to show that two triangles are similar.

Is $\triangle ABC \sim \triangle DBF$? How do you know?



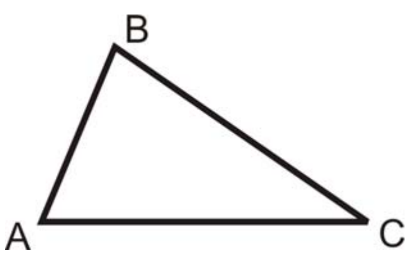
Is $\triangle ABC \sim \triangle DEC$? How do you know?



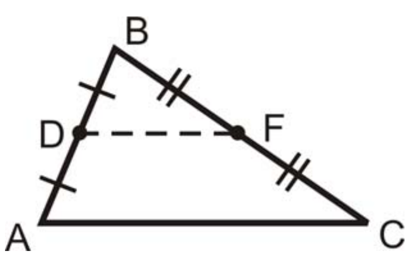
Review

A **midsegment** is a line segment that connects two midpoints of adjacent sides of a triangle. For every triangle there are three midsegments.

Draw the midsegment $\overline{DF}$ between $\overline{AB}$ and $\overline{BC}$. Use appropriate tic marks.

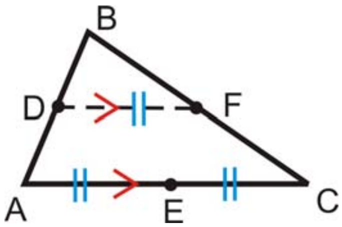


Find the midpoints of $\overline{AB}$ and $\overline{BC}$ using your ruler. Label these points D and F. Connect them to create the midsegment.



Don't forget to put the tic marks, indicating that D and F are midpoints, $\overline{AD} \cong \overline{DB}$ and $\overline{BF} \cong \overline{FC}$.

There are two important properties of midsegments that combine to make the **Midsegment Theorem**. The **Midsegment Theorem** states that the midsegment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle, and the length of this midsegment is half the length of the third side. So, if $\overline{DF}$ is a midsegment of $\triangle ABC$, then $DF = \frac{1}{2}AC = AE = EC$ and $\overline{DF} \| \overline{AC}$.



Note that there are two important ideas here. One is that the midsegment is parallel to a side of the triangle. The other is that the midsegment is always half the length of this side. To play with the properties of midsegments, go to http://www.mathopenref.com/trianglemidsegment.html .

Watch This

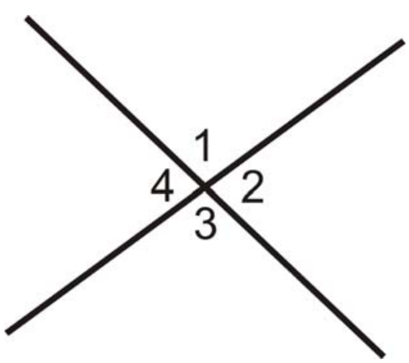
MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/54526

CK-12 Midsegment Theorem

Vertical angles are two non-adjacent angles formed by intersecting lines. In the picture below, $\angle 1$ and $\angle 3$ are vertical angles and $\angle 2$ and $\angle 4$ are vertical angles.



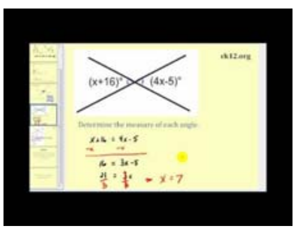
Vertical Angles Theorem: If two angles are vertical angles, then they are congruent.

We can prove the Vertical Angles Theorem using the same process we used above. However, let's not use any specific values for the angles.

From the picture above:	
$\angle 1$ and $\angle 2$ are a linear pair	$m\angle 1 + m\angle 2 = 180^\circ$
$\angle 2$ and $\angle 3$ are a linear pair	$m\angle 2 + m\angle 3 = 180^\circ$
$\angle 3$ and $\angle 4$ are a linear pair	$m\angle 3 + m\angle 4 = 180^\circ$
All of the equations $= 180^\circ$, so set the first and second equation equal to each other and the second and third.	$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$ *AND* $m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$
Cancel out the like terms	$m\angle 1 = m\angle 3,\ m\angle 2 = m\angle 4$

Recall that anytime the measures of two angles are equal, the angles are also congruent.

Watch This



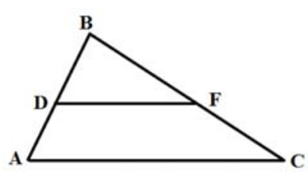
MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/1274

James Sousa: Vertical Angles

Example A



Given: $\overline{DF}$ is the midsegment of $\triangle ABC$

Prove: $\triangle ABC \sim \triangle DBF$

TABLE 7.1:

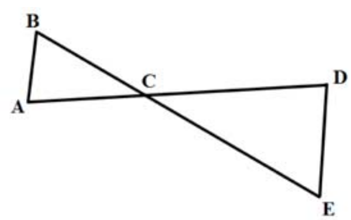
Statement	Reason
1. $\overline{DF}$ is the midsegment of $\triangle ABC$	2. Given
3.	4. Midsegment Theorem
5.	6.
7.	8.
9. $\triangle ABC \sim \triangle DBF$	10.

Answer:

TABLE 7.2:

Statement	Reason
1. $\overline{DF}$ is the midsegment of $\triangle ABC$	2. Given
3. $\overline{AC} \| \overline{DF}$	4. Midsegment Theorem
5. $\angle BAC \cong \angle BDF$	6. Corresponding Angles Postulate
7. $\angle BCA \cong \angle BFD$	8. Corresponding Angles Postulate
9. $\triangle ABC \sim \triangle DBF$	10. AA Similarity Postulate

Example B



Given: $\frac{BC}{EC} = \frac{AC}{DC}$

Prove: $\triangle ABC \sim \triangle DEC$

TABLE 7.3:

Statement	Reason
1. $\frac{BC}{EC} = \frac{AC}{DC}$	2. Given
3.	4.
5. $\triangle ABC \sim \triangle DEC$	6.

Answer:

TABLE 7.4:

Statement	Reason
1. $\frac{BC}{EC} = \frac{AC}{DC}$	2. Given
3. $\angle BCA \cong \angle ECD$	4. Vertical Angles Theorem
5. $\triangle ABC \sim \triangle DEC$	6. SAS Similarity Theorem

Watch This

Prove that two triangles are similar by writing an algebraic proof. https://learnzillion.com/lessons/2787-prove-that-two-triangles-are-similar-by-writing-an-algebraic-proof

Vocabulary

A line segment that connects two midpoints of the sides of a triangle is called a ***midsegment***. A ***midpoint*** is a point that divides a segment into two equal pieces. Two lines are ***parallel*** if they never intersect. Parallel lines have slopes that are equal. In a triangle, midsegments are always parallel to one side of the triangle.

CONCEPT 8 SLT 10 Use triangle similarity to prove the Triangle Proportionality Theorem and its converse.

What if you were given a triangle with a line segment drawn through it from one side to the other? How could you use information about the triangle's side lengths to determine if that line segment is parallel to the third side? How does this relate to similarity?

Watch This

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1354

James Sousa: Triangle Proportionality Theorem

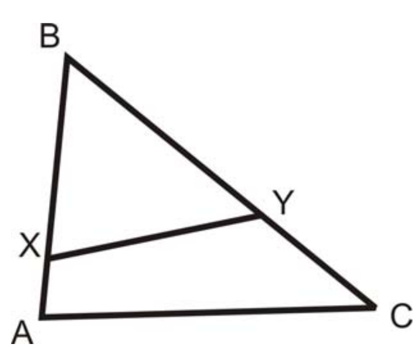
Guidance

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle and divides the other two sides into congruent halves. The midsegment divides those two sides **proportionally.**

Investigation: Triangle Proportionality

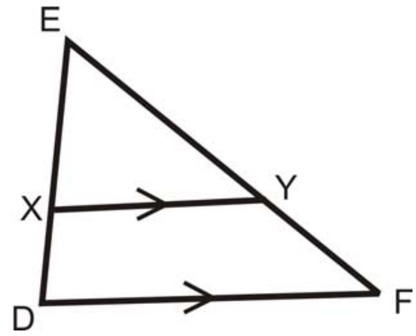
Tools Needed: pencil, paper, ruler

1. Draw $\triangle ABC$. Label the vertices.
2. Draw $\overline{XY}$ so that X is on $\overline{AB}$ and Y is on $\overline{BC}$. X and Y can be ***anywhere*** on these sides.



3. Is $\triangle XBY \sim \triangle ABC$? Why or why not? Measure AX, XB, BY, and YC. Then set up the ratios $\frac{AX}{XB}$ and $\frac{YC}{YB}$. Are they equal?

4. Draw a second triangle, $\triangle DEF$. Label the vertices.
5. Draw $\overline{XY}$ so that X is on $\overline{DE}$ and Y is on $\overline{EF}$ AND $\overline{XY} \parallel \overline{DF}$.
6. Is $\triangle XEY \sim \triangle DEF$? Why or why not? Measure DX, XE, EY, and YF. Then set up the ratios $\frac{DX}{XE}$ and $\frac{FY}{YE}$. Are they equal?

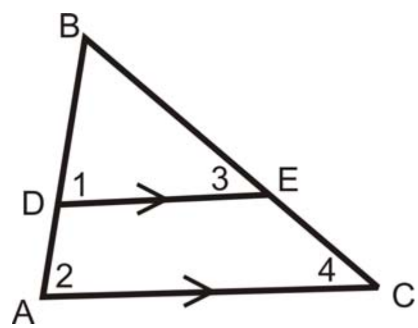


From this investigation, it is clear that if the line segments are parallel, then $\overline{XY}$ divides the sides proportionally.

Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof of the Triangle Proportionality Theorem:



Given: $\triangle ABC$ with $\overline{DE} \parallel \overline{AC}$

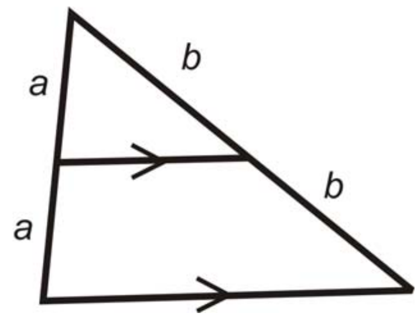
Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

TABLE 8.1:

Statement	*Reason*
1. $\overline{DE} \parallel \overline{AC}$	Given
2. $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$	Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DBE$	AA Similarity Postulate
4. $AD + DB = AB, EC + EB = BC$	Segment Addition Postulate
5. $\frac{AB}{BD} = \frac{BC}{BE}$	Corresponding sides in similar triangles are proportional
6. $\frac{AD+DB}{BD} = \frac{EC+EB}{BE}$	Substitution PoE
7. $\frac{AD}{BD} + \frac{DB}{DB} = \frac{EC}{BE} + \frac{BE}{BE}$	Separate the fractions
8. $\frac{AD}{BD} + 1 = \frac{EC}{BE} + 1$	Substitution PoE (something over itself always equals 1)
9. $\frac{AD}{BD} = \frac{EC}{BE}$	Subtraction PoE

Example A

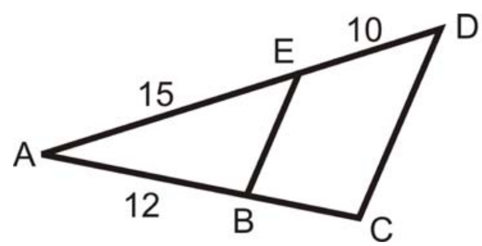
A triangle with its midsegment is drawn below. What is the ratio that the midsegment divides the sides into?



The midsegment's endpoints are the midpoints of the two sides it connects. The midpoints split the sides evenly. Therefore, the ratio would be $a:a$ or $b:b$. Both of these reduce to 1:1.

Example B

In the diagram below, $\overline{EB} \,||\, \overline{CD}$. Find BC.

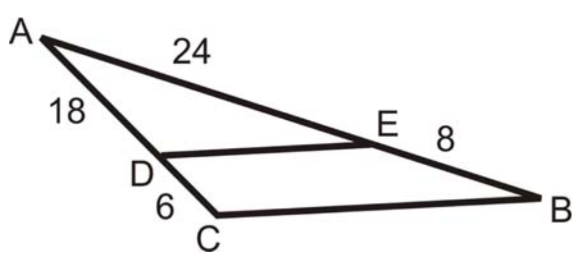


Use the Triangle Proportionality Theorem.

$$\frac{10}{15} = \frac{BC}{12} \longrightarrow 15(BC) = 120$$
$$BC = 8$$

Example C

Is $\overline{DE} \,||\, \overline{CB}$?



Use the Triangle Proportionality Converse. If the ratios are equal, then the lines are parallel.

$\frac{6}{18} = \frac{1}{3}$ and $\frac{8}{24} = \frac{1}{3}$

Because the ratios are equal, $\overline{DE} \parallel \overline{CB}$.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52551

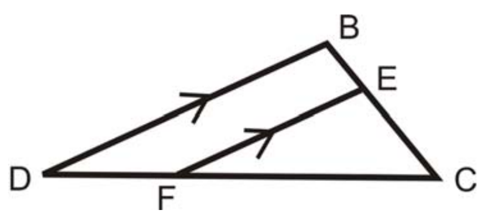
CK-12 Foundation: Chapter7TriangleProportionalityB

Vocabulary

A line segment that connects two midpoints of the sides of a triangle is called a ***midsegment***. A midpoint is a point that divides a segment into two equal pieces. Pairs of numbers are ***proportional*** if they are in the same ratio.

Guided Practice

Use the diagram to answers questions 1-5. $\overline{DB} \| \overline{FE}$.



1. Name the similar triangles. Write the similarity statement.

2. $\frac{BE}{EC} = \frac{?}{FC}$

3. $\frac{EC}{CB} = \frac{CF}{?}$

4. $\frac{DB}{?} = \frac{BC}{EC}$

5. $\frac{FC+?}{FC} = \frac{?}{FE}$

Answers:

1. $\triangle DBC \sim \triangle FEC$

2. DF

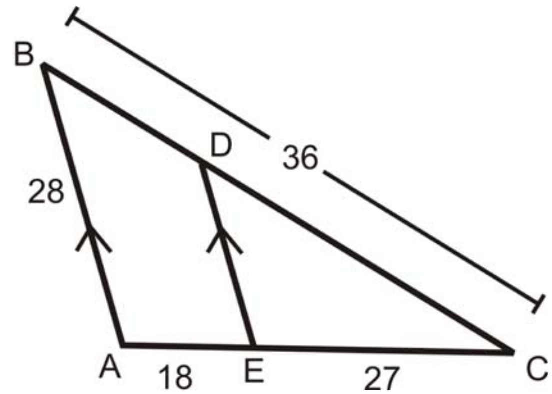
3. DC

4. FE

5. DF; DB

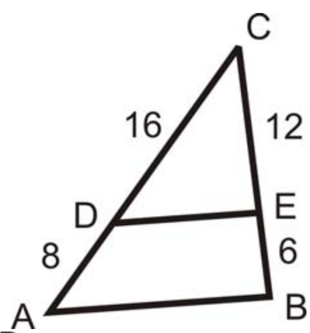
Explore More

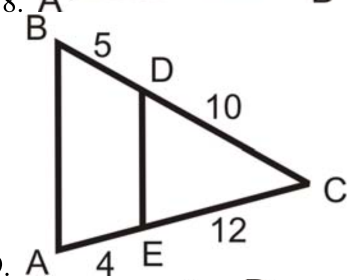
Use the diagram to answer questions 1-7. $\overline{AB} \parallel \overline{DE}$.

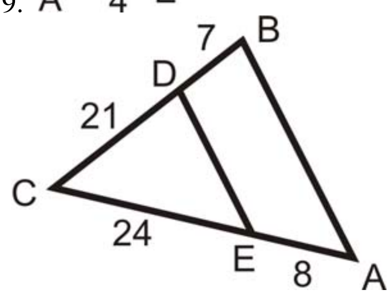


1. Find BD.
2. Find DC.
3. Find DE.
4. Find AC.
5. What is $BD : DC$?
6. What is $DC : BC$?
7. We know that $\frac{BD}{DC} = \frac{AE}{EC}$ and $\frac{BA}{DE} = \frac{BC}{DC}$. Why is $\frac{BA}{DE} \neq \frac{BD}{DC}$?

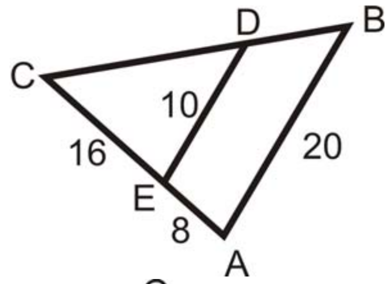
Use the given lengths to determine if $\overline{AB} \parallel \overline{DE}$.

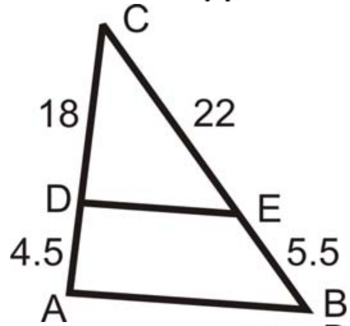
8. 

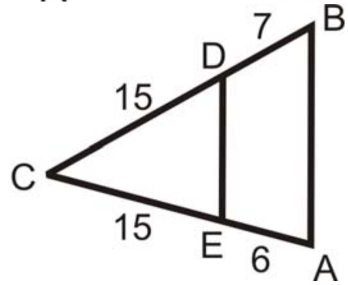
9. 

10. 

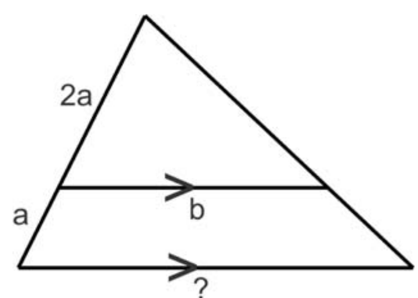
 SLT 10 Use triangle similarity to prove the Triangle Proportionality Theorem and its converse.

11. 

12. 

13. 

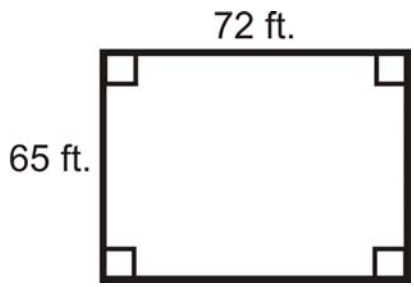
Find the unknown length.

14. 

15. What is the ratio that the midsegment divides the sides into?

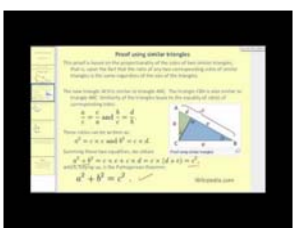
CONCEPT 9 SLT 11 Use triangle similarity to prove the Pythagorean Theorem and its converse.

What if a friend of yours wanted to design a rectangular building with one wall 65 ft long and the other wall 72 ft long? How can he ensure the walls are going to be perpendicular? After completing this Concept, you'll be able to apply the Pythagorean Theorem in order to solve problems like these.



Watch This

This video demonstrates the proof of the Pythagorean Theorem using triangle similarity.

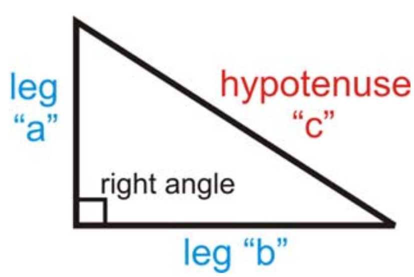


MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1372

James Sousa: Pythagorean Theorem

Guidance

The sides of a right triangle are called legs (the sides of the right angle) and the side opposite the right angle is the hypotenuse. For the Pythagorean Theorem, the legs are "a" and "b" and the hypotenuse is "c".



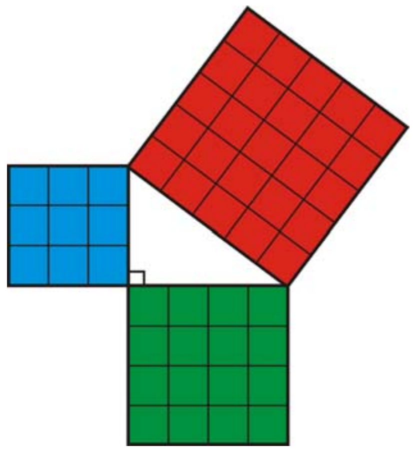
Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, then $a^2 + b^2 = c^2$.

The video above proves the Pythagorean Theorem using similar triangles. However, there are several proofs of the Pythagorean Theorem, one of which is shown below.

Investigation: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in square and a right triangle with legs of 3 and 4 inches.
2. Cut out the triangle and square and arrange them like the picture on the right.

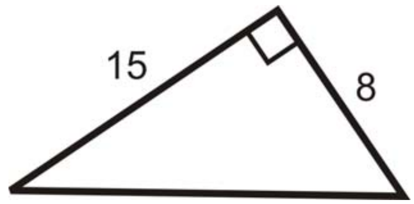


3. This theorem relies on area. Recall from a previous math class, that the area of a square is length times width. But, because the sides are the same you can rewrite this formula as $A_{square} = length \times width = side \times side = side^2$. So, the Pythagorean Theorem can be interpreted as $(square\ with\ side\ a)^2 + (square\ with\ side\ b)^2 = (square\ with\ side\ c)^2$. In this Investigation, the sides are 3, 4 and 5 inches. What is the area of each square?
4. Now, we know that $9 + 16 = 25$, or $3^2 + 4^2 = 5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

For two animated proofs, go to http://www.mathsisfun.com/pythagoras.html and scroll down to "And You Can Prove the Theorem Yourself."

Example A

Find the length of the hypotenuse of the triangle below.



Let's use the Pythagorean Theorem. Set a and b equal to 8 and 15 and solve for c, the hypotenuse.

$$\begin{aligned} 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 && \textit{Take the square root of both sides.} \\ 17 &= c \end{aligned}$$

When you take the square root of an equation, usually the answer is +17 or -17. Because we are looking for length, we only use the positive answer. ***Length is never negative***.

Concept Problem Revisited

To make the walls perpendicular, find the length of the diagonal.

$$\begin{aligned} 65^2 + 72^2 &= c^2 \\ 4225 + 5184 &= c^2 \\ 9409 &= c^2 \\ 97 &= c \end{aligned}$$

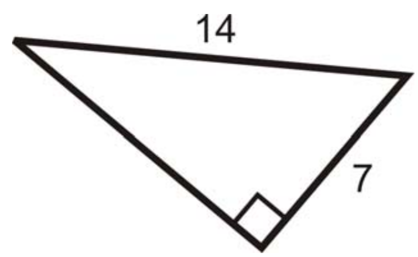
In order to make the building rectangular, both diagonals must be 97 feet.

Vocabulary

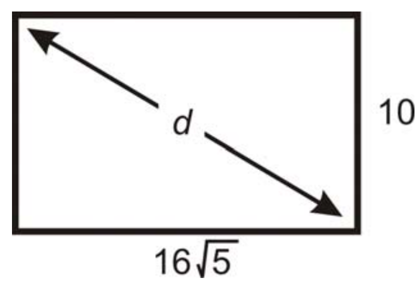
The two shorter sides of a right triangle (the sides that form the right angle) are the ***legs*** and the longer side (the side opposite the right angle) is the ***hypotenuse***. The ***Pythagorean Theorem*** states that $a^2 + b^2 = c^2$, where the legs are "a" and "b" and the hypotenuse is "c". A combination of three numbers that makes the Pythagorean Theorem true is called a ***Pythagorean triple.***

Guided Practice

1. Find the missing side of the right triangle below.



2. What is the diagonal of a rectangle with sides 10 and $16\sqrt{5}$?



Answers:

1. Here, we are given the hypotenuse and a leg. Let's solve for b.

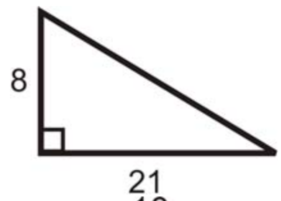
$$\begin{aligned} 7^2+b^2 &= 14^2 \\ 49+b^2 &= 196 \\ b^2 &= 147 \\ b &= \sqrt{147} = \sqrt{7\cdot 7\cdot 3} = 7\sqrt{3} \end{aligned}$$

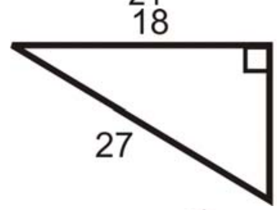
2. For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d.

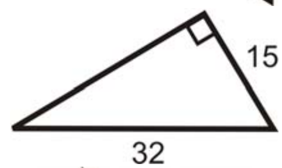
$$\begin{aligned} 10^2+\left(16\sqrt{5}\right)^2 &= d^2 \\ 100+1280 &= d^2 \\ 1380 &= d^2 \\ d &= \sqrt{1380} = 2\sqrt{345} \end{aligned}$$

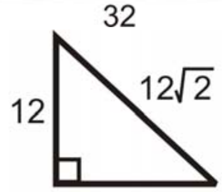
Explore More

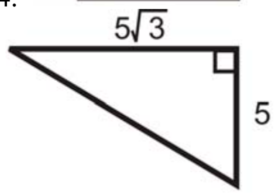
Find the length of the missing side. Simplify all radicals.

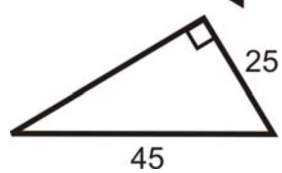
1. 

2. 

3. 

4. 

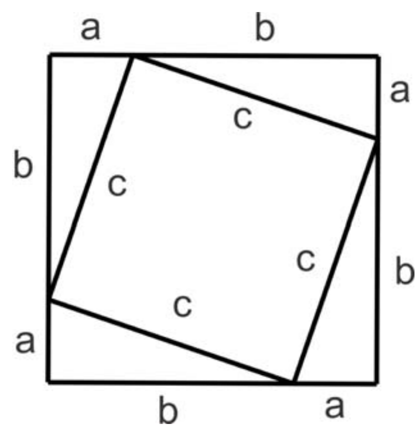
5. 

6. 

7. If the legs of a right triangle are 10 and 24, then the hypotenuse is _____________.
8. If the sides of a rectangle are 12 and 15, then the diagonal is _____________.
9. If the legs of a right triangle are x and y, then the hypotenuse is _____________.
10. If the sides of a square are 9, then the diagonal is _____________.

More Pythagorean Theorem Proofs

The first proof below is similar to the one done earlier in this Concept. This proof was explored in C2.0 Algebra 1 as well. Use the picture below to answer the following questions.

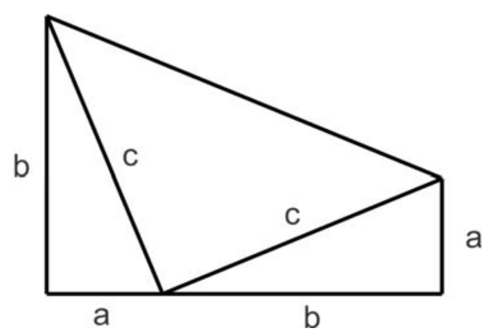


11. Find the area of the square with sides $(a+b)$.

12. Find the sum of the areas of the square with sides c and the right triangles with legs a and b.

13. The areas found in #11 and #12 should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

Major General James A. Garfield (and former President of the U.S) is credited with deriving this next proof of the Pythagorean Theorem using a trapezoid.



14. Find the area of the trapezoid using the trapezoid area formula: $A = \frac{1}{2}(b_1 + b_2)h$

15. Find the sum of the areas of the three right triangles in the diagram.

11. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

CONCEPT 10 SLT 12 Use triangle similarity to prove the Pythagorean Theorem and its converse.

What if you were told that a triangle had side lengths of 5, 12, and 13? How could you determine if the triangle were a right one? After completing this Concept, you'll be able to use the Pythagorean Theorem to solve problems like this one.

Watch This

MEDIA

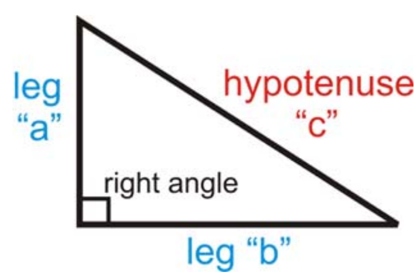
Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/80484

CK-12 Foundation: Using The Pythagorean Theorem

Review

The two shorter sides of a right triangle (the sides that form the right angle) are the **legs** and the longer side (the side opposite the right angle) is the **hypotenuse**. For the Pythagorean Theorem, the legs are "*a*" and "*b*" and the hypotenuse is "*c*".



Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, $a^2+b^2=c^2$.

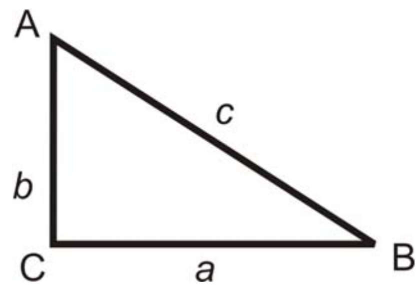
For proofs of the Pythagorean Theorem go to: http://www.mathsisfun.com/pythagoras.html and scroll down to "And You Can Prove the Theorem Yourself."

Guidance

The converse of the Pythagorean Theorem is also true. It allows you to prove that a triangle is a right triangle even if you do not know its angle measures.

Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

If $a^2+b^2=c^2$, then $\triangle ABC$ is a right triangle.



Pythagorean Triples

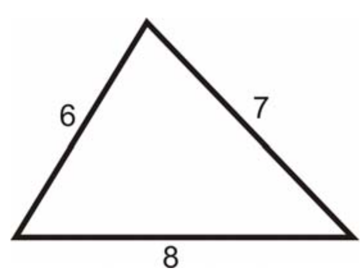
A combination of three numbers that makes the Pythagorean Theorem true is called a **Pythagorean triple.** Each set of numbers below is a Pythagorean triple.

$$3,4,5 \qquad 5,12,13 \qquad 7,24,25 \qquad 8,15,17 \qquad 9,12,15 \qquad 10,24,26$$

Any multiple of a Pythagorean triple is also considered a Pythagorean triple. Multiplying 3, 4, 5 by 2 gives 6, 8, 10, which is another triple. To see if a set of numbers makes a Pythagorean triple, plug them into the Pythagorean Theorem.

Example A

Do 6, 7, and 8 make the sides of a right triangle?



Plug the three numbers into the Pythagorean Theorem. Remember that the largest length will always be the hypotenuse, c. If $6^2+7^2=8^2$, then they are the sides of a right triangle.

$$6^2+7^2=36+49=85$$
$$8^2=64 \qquad 85 \neq 64, \text{ so the lengths are not the sides of a right triangle.}$$

Example B

Is 20, 21, 29 a Pythagorean triple?

If $20^2+21^2=29^2$, then the set is a Pythagorean triple.

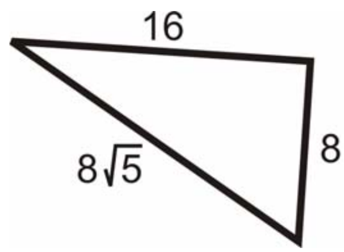
$$\begin{aligned} 20^2+21^2 &= 400+441=841 \\ 29^2 &= 841 \end{aligned}$$

Therefore, 20, 21, and 29 is a Pythagorean triple.

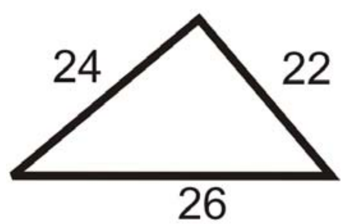
Example C

Determine if the triangles below are right triangles.

a)



b)



Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest side represent c.

a)

$$\begin{aligned} a^2+b^2 &= c^2 \\ 8^2+16^2 &\stackrel{?}{=} \left(8\sqrt{5}\right)^2 \\ 64+256 &\stackrel{?}{=} 64\cdot 5 \\ 320 &= 320 \qquad \text{Yes} \end{aligned}$$

b)

$$\begin{aligned} a^2+b^2 &= c^2 \\ 22^2+24^2 &\stackrel{?}{=} 26^2 \\ 484+576 &\stackrel{?}{=} 676 \\ 1060 &\neq 676 \qquad \text{No} \end{aligned}$$

Guided Practice

1. Do the following lengths make a right triangle?

a) $\sqrt{5}, 3, \sqrt{14}$

b) $6, 2\sqrt{3}, 8$

c) $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Answers:

1. Even though there is no picture, you can still use the Pythagorean Theorem. Again, the longest length will be c.

a)

$$\begin{aligned}&\left(\sqrt{5}\right)^2 + 3^2 = \sqrt{14}^2\\&5 + 9 = 14\\&\text{Yes}\end{aligned}$$

b)

$$\begin{aligned}6^2 + \left(2\sqrt{3}\right)^2 &= 8^2\\36 + (4 \cdot 3) &= 64\\36 + 12 &\neq 64\end{aligned}$$

c) This is a multiple of $\sqrt{2}$ of a 3, 4, 5 right triangle. Yes, this is a right triangle.

Practice

Determine if the following sets of numbers are Pythagorean Triples.

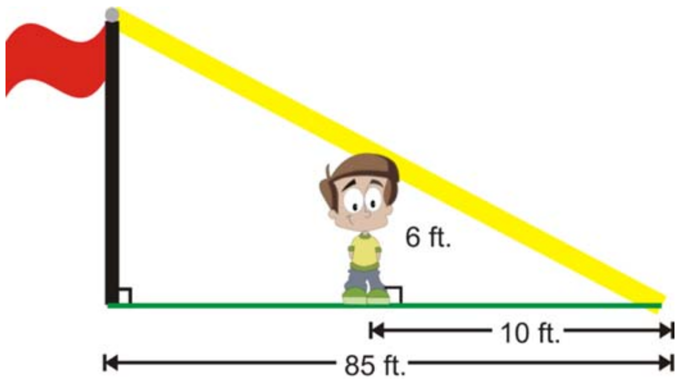
2. 9, 17, 18
3. 10, 15, 21
4. 11, 60, 61
5. 15, 20, 25
6. 18, 73, 75

Determine if the following lengths make a right triangle.

7. 7, 24, 25
8. $\sqrt{5}, 2\sqrt{10}, 3\sqrt{5}$
9. $2\sqrt{3}, \sqrt{6}, 8$
10. 15, 20, 25
11. 20, 25, 30
12. $8\sqrt{3}, 6, 2\sqrt{39}$

CONCEPT 11 SLT 13 & 14 Apply triangle congruence and triangle similarity to solve problem situations.

What if you wanted to measure the height of a flagpole using your friend George? He is 6 feet tall and his shadow is 10 feet long. At the same time, the shadow of the flagpole was 85 feet long. How tall is the flagpole?



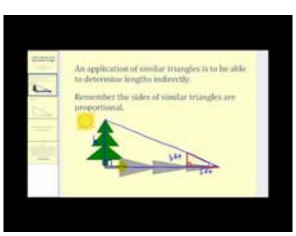
Watch This

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/52558

CK-12 Foundation: Chapter7IndirectMeasurementA

MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/132331

http://www.youtube.com/watch?v=31qq1zoQVHY



MEDIA
Click image to the left for use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/1345

James Sousa: Indirect Measurement Using Similar Triangles

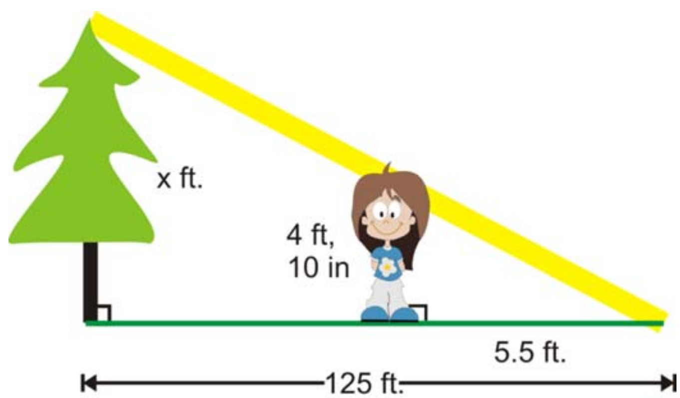
Guidance

An application of similar triangles is to measure lengths *indirectly*. The length to be measured would be some feature that was not easily accessible to a person, such as the width of a river or canyon and the height of a tall object. To measure something indirectly, you need to set up a pair of similar triangles.

Example A

A tree outside Ellie's building casts a 125 foot shadow. At the same time of day, Ellie casts a 5.5 foot shadow. If Ellie is 4 feet 10 inches tall, how tall is the tree?

Draw a picture. From the picture to the right, we see that the tree and Ellie are parallel, therefore the two triangles are similar to each other. Write a proportion.



$$\frac{4ft, 10in}{xft} = \frac{5.5ft}{125ft}$$

Notice that our measurements are not all in the same units. Change both numerators to inches and then we can cross multiply.

$$\begin{aligned}\frac{58in}{xft} = \frac{66in}{125ft} \longrightarrow 58(125) &= 66(x)\\ 7250 &= 66x\\ x &\approx 109.85\ ft\end{aligned}$$

Example B

Cameron is 5 ft tall and casts a 12 ft shadow. At the same time of day, a nearby building casts a 78 ft shadow. How tall is the building?

To solve, set up a proportion that compares height to shadow length for Cameron and the building. Then solve the equation to find the height of the building. Let x represent the height of the building.

$$\frac{5ft}{12ft} = \frac{x}{78ft}$$
$$12x = 390$$
$$x = 32.5ft$$

The building is 32.5 feet tall.

Example C

The Empire State Building is 1250 ft. tall. At 3:00, Pablo stands next to the building and has an 8 ft. shadow. If he is 6 ft tall, how long is the Empire State Building's shadow at 3:00?

Similar to Example B, solve by setting up a proportion that compares height to shadow length. Then solve the equation to find the length of the shadow. Let x represent the length of the shadow.

$$\frac{6ft}{8ft} = \frac{1250ft}{x}$$
$$6x = 10000$$
$$x = 1666.67ft$$

The shadow is approximately 1666.67 feet long.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/52559

CK-12 Foundation: Chapter7IndirectMeasurementB

Concept Problem Revisited

It is safe to assume that George and the flagpole stand vertically, making right angles with the ground. Also, the angle where the sun's rays hit the ground is the same for both. The two trianglesare similar. Set up a proportion.

$$\frac{10}{85} = \frac{6}{x} \longrightarrow 10x = 510$$
$$x = 51\ ft.$$

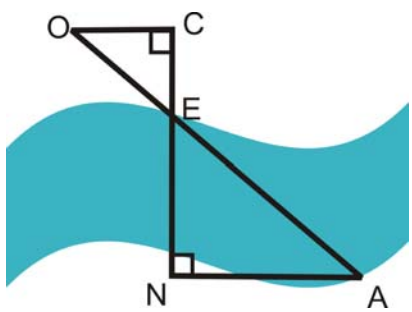
The height of the flagpole is 51 feet.

Vocabulary

Two triangles are ***similar*** if all their corresponding angles are ***congruent*** (exactly the same) and their corresponding sides are ***proportional*** (in the same ratio). Solve proportions by ***cross-multiplying***.

Guided Practice

In order to estimate the width of a river, the following technique can be used. Use the diagram.



Place three markers, O, C, and E on the upper bank of the river. E is on the edge of the river and $\overline{OC} \perp \overline{CE}$. Go across the river and place a marker, N so that it is collinear with C and E. Then, walk along the lower bank of the river and place marker A, so that $\overline{CN} \perp \overline{NA}$. $OC = 50$ feet, $CE = 30$ feet, $NA = 80$ feet.

1. Is $\triangle OCE \sim \triangle ANE$? How do you know?

2. Is $\overline{OC} \| \overline{NA}$? How do you know?

3. What is the width of the river? Find EN.

Answers:

1. Yes. $\angle C \cong \angle N$ because they are both right angles. $\angle OEC \cong \angle AEN$ because they are vertical angles. This means $\triangle OCE \sim \triangle ANE$ by the AA Similarity Postulate.

2. Since the two triangles are similar, we must have $\angle EOC \cong \angle EAN$. These are alternate interior angles. When alternate interior angles are congruent then lines are parallel, so $\overline{OC} \| \overline{NA}$.

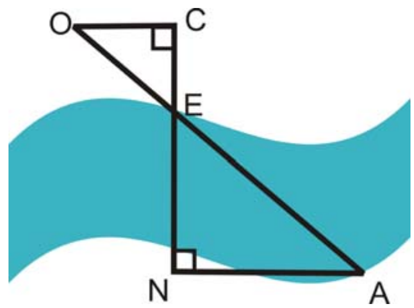
3. Set up a proportion and solve by cross-multiplying.

$$\frac{30ft}{EN} = \frac{50ft}{80ft}$$
$$50(EN) = 2400$$
$$EN = 48$$

The river is 48 feet wide.

Explore More

The technique from the guided practice section was used to measure the distance across the Grand Canyon. Use the picture below and $OC = 72\ ft, CE = 65\ ft$, and $NA = 14,400\ ft$ for problems 1 - 3.



1. Find EN (the distance across the Grand Canyon).
2. Find OE.
3. Find EA.

4. Janet wants to measure the height of her apartment building. She places a pocket mirror on the ground 20 ft from the building and steps backwards until she can see the top of the build in the mirror. She is 18 in from the mirror and her eyes are 5 ft 3 in above the ground. The angle formed by her line of sight and the ground is congruent to the angle formed by the reflection of the building and the ground. You may wish to draw a diagram to illustrate this problem. How tall is the building?
5. Sebastian is curious to know how tall the announcer's box is on his school's football field. On a sunny day he measures the shadow of the box to be 45 ft and his own shadow is 9 ft. Sebastian is 5 ft 10 in tall. Find the height of the box.
6. Juanita wonders how tall the mast of a ship she spots in the harbor is. The deck of the ship is the same height as the pier on which she is standing. The shadow of the mast is on the pier and she measures it to be 18 ft long. Juanita is 5 ft 4 in tall and her shadow is 4 ft long. How tall is the ship's mast?
7. Evan is 6 ft tall and casts a 15 ft shadow. A the same time of day, a nearby building casts a 30 ft shadow. How tall is the building?
8. Priya and Meera are standing next to each other. Priya casts a 10 ft shadow and Meera casts an 8 ft shadow. Who is taller? How do you know?
9. Billy is 5 ft 9 inches tall and Bobby is 6 ft tall. Bobby's shadow is 13 feet long. How long is Billy's shadow?
10. Sally and her little brother are walking to school. Sally is 4 ft tall and has a shadow that is 3 ft long. Her little brother's shadow is 2 ft long. How tall is her little brother?
11. Ray is outside playing basketball. He is 5 ft tall and at this time of day is casting a 12 ft shadow. The basketball hoop is 10 ft tall. How long is the basketball hoop's shadow?
12. Jack is standing next to a very tall tree and wonders just how tall it is. He knows that he is 6 ft tall and at this moment his shadow is 8 ft long. He measures the shadow of the tree and finds it is 90 ft. How tall is the tree?
13. Jason, who is 4 ft 9 inches tall is casting a 6 ft shadow. A nearby building is casting a 42 ft shadow. How tall is the building?
14. Alexandra, who is 5 ft 8 in tall is casting a 12 ft shadow. A nearby lamppost is casting a 20 ft shadow. How tall is the lamppost?
15. Use shadows or a mirror to measure the height of an object in your yard or on the school grounds. Draw a picture to illustrate your method.